Math 6510 Homework 10

Tarun Chitra

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§2.2 Problems

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Problem. *Compute the homology group of the following 2-complexes X:*
a) The quotient of S^2 obtained by identifying north and south poles to a point
b) $S^1 \times (S^1 \vee S^1)$
c) The space obtained from D^2 by first deleting the interiors of two disjoint subdisks in the interior of D^2 and then identifying all three resulting boundary circles together via homeomorphisms preserving clockwise orientation of the circles
d) The quotient space of $S^1 \times S^1$ obtained by identifying points in the circle $S^1 \times \{x_0\}$ that differ by a $\frac{2\pi}{m}$ rotation and identifying points in the circle $\{x_0\} \times S^1$ that differ by $\frac{2\pi}{n}$ rotation

a)

From Example 0.8, $X \cong S^2 \vee S^1$ so that Corollary 2.25 gives,

$$H_n(X) \cong H_n(S^2 \vee S^1) = \begin{cases} \mathbb{Z} & n \in \{0,1,2\} \\ 0 & \text{Otherwise} \end{cases} \tag{1}$$

b)

Let's first establish what the $H_\bullet(X)$ is via the Künneth Formula and then use cellular homology to verify. Firstly note that for all $n \in \mathbb{N} \cup \{0\}$,

$$\mathbf{Tor}^{\mathbb{Z}}\left(H_n(S^1), H_n(S^1)\right) = \mathbf{Tor}^{\mathbb{Z}}\left(H_n(S^1), H_n(S^1 \vee S^1\right) = \mathbf{Tor}^{\mathbb{Z}}\left(H_n(S^1 \vee S^1), H_n(S^1 \vee S^1)\right) = 0 \tag{2}$$

since the only non-trivial groups are $H_1(S^1) \cong \mathbb{Z}, H_1(S^1, S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$ [via Corollary 2.25] which are both free abelian. Hence the Künneth formula reduces to the exact sequence,

$$0 \to \bigoplus_{i+j=k} H_i(S^1, \mathbb{Z}) \otimes_{\mathbb{Z}} H_i(S^1 \vee S^1, \mathbb{Z}) \to H_k(S^1 \times (S^1 \vee S^1), \mathbb{Z}) \to 0 \tag{3}$$

Hence, $H_k(S^1 \times (S^1 \vee S^1), \mathbb{Z}) \cong \bigoplus_{i+j=k} H_i(S^1, \mathbb{Z}) \otimes_{\mathbb{Z}} H_i(S^1 \vee S^1, \mathbb{Z})$ giving us:

$$\begin{aligned} H_0(S^1 \times (S^1 \vee S^1), \mathbb{Z}) &\cong \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z} \\ H_1(S^1 \times (S^1 \vee S^1), \mathbb{Z}) &\cong \mathbb{Z} \times \mathbb{Z} \oplus \mathbb{Z} \otimes (\mathbb{Z} \oplus \mathbb{Z}) \cong \mathbb{Z}^3 \\ H_2(S^1 \times (S^1 \vee S^1), \mathbb{Z}) &\cong \mathbb{Z} \otimes (\mathbb{Z} \oplus \mathbb{Z}) \cong \mathbb{Z}^2 \end{aligned} \tag{4}$$

where the final inequalities come from the fact that the tensor product abelianizes the products.

Given this algebraic result, let's verify it geometrically. Since $S^1 \times S^1$ is a torus, the geometric picture for $S^1 \times (S^1 \vee S^1)$ is very similar and in fact, we can represent it as a mapping torus of a map $g : S^1 \to S^1 \vee S^1$. From Example 2.48, we see that if $f, g : X' \to X'$ are $\mathbb{1}_{X'}$, then the mapping cylinder $Z = X' \times I / \sim$ is homeomorphic to $X' \times S^1$. In this case, letting $X' = S^1 \vee S^1$, we get the exact sequence,

$$\cdots \longrightarrow H_n(S^1 \vee S^1) \xrightarrow{0} H_n(S^1 \vee S^1) \longrightarrow H_n(X) \longrightarrow H_{n-1}(S^1 \vee S^1) \to \cdots$$

Note that the map that is trivial is due to the fact that $f = g$ so $f_* = g_*$ in the augmented Mayer-Vietoris Sequence of Example 2.48. Now since $H_2(S^1 \vee S^1) = 0$, we have the exact sequence,

$$0 \longrightarrow H_2(X) \longrightarrow H_1(S^1 \vee S^1) \longrightarrow 0$$

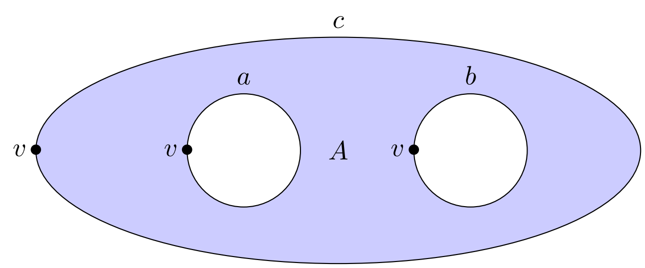
so that $H_2(X) \cong H_1(S^1 \vee S^1) \cong \mathbb{Z}^2$. On the other hand for $n = 1$ we have the short exact sequence,

$$0 \longrightarrow H_1(S^1 \vee S^1) \longrightarrow H_1(X) \longrightarrow H_0(S^1 \vee S^1) \to 0$$

which becomes the short exact sequence $0 \to \mathbb{Z}^2 \to H_1(X) \to \mathbb{Z} \to 0$ which implies that $H_1(X) \cong \mathbb{Z}^3$. Finally, it is clear that X is connected so $H_0(X) \cong \mathbb{Z}$. Hence we've verified the algebraic result.

c)

We will place the following CW structure on X with 1 0-cell v, 3 1-cells a, b, c and 1 2-cell A:



Associated to this, we have the following chain complex,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

Firstly, it is clear that X is path-connected so $H_0(X) \cong \mathbb{Z}$. Now $\ker d_1 = \mathbb{Z}^3$ since the boundaries of all of the 1-cells are trivial. Now A is attached via the word $[a, b]ca^{-1}c^{-1}$ so that after abelianianization, $d_2 A = -a$. Hence $\operatorname{Im} d_2 = \langle a \rangle$ so $H_1(X) \cong \mathbb{Z}$. Exactness implies that $\ker d_2 = 0$ so that we have:

$$H_k(X) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}^2 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

d)

We can start with the 1-skeleton for a torus, namely 1 0-cell v and 2 1-cells a, b arranged in the form of $S^1 \vee S^1$. The difference here is that we now attach the 2-cell A via the word $a^n b^m a^{-n} b^{-m}$ in order to preserve the quotient. We have the cell complex,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

This space is again clearly path-connected so $H_0(X) \cong \mathbb{Z}$. Now all of the 1-cells end and begin on v, so $\ker d_1 = \mathbb{Z}^2$. From the attaching word, we have $d_2 = 0$ so $H_1(X) \cong \mathbb{Z}^2, H_2(X) = \mathbb{Z}$. Summary:

$$H_k(X) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^2 & k = 1 \\ \mathbb{Z} & k = 2 \\ 0 & \text{else} \end{cases}$$

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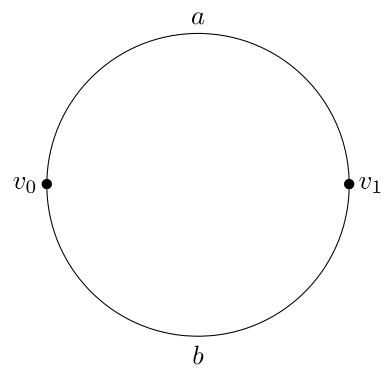
Problem. *Let X be the quotient space of S^2 under the identifications $x \sim -x$ for x in the equator S^1. Compute the homology groups $H_i(X)$. Do the same for S^3 with the antipodal points of the equatorial $S^2 \subset S^3$ identified*

In the case of $X = S^2/\sim$, we give it the CW structure with 2 0-cells, $\{v_0, v_1\}$, 2 1-cells, $\{a, b\}$, and 2 2-cells, $\{A, B\}$, where the one skeleton is of the form,



We glue the 2-cell A along the word ab and the 2-cell B along the word $a^{-1}b^{-1}$. Under the quotient $a = b, v_0 = v_1$, so the 2-cells are glued along $2a, -2a$, respectively. Our chain complex is,

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1=0} \mathbb{Z} \longrightarrow 0$$

The space is path-connected so we have $H_0(X) \cong \mathbb{Z}$. Now $\ker d_1 = \mathbb{Z}$ since $d_1 = 0$ and from the attaching map, $\operatorname{Im} d_2 = \langle 2a \rangle$. Hence $H_1(X) \cong \mathbb{Z}_2$. As such, we have $H_2(X) \cong \mathbb{Z}$. Summary:

$$H_k(X) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2 \\ \mathbb{Z}_2 & \text{if } k = 1 \\ 0 & \text{else} \end{cases}$$

In the case of $Y = S^3/\sim$, we give the same CW structure with two k-cells for $k \in \{0, 1, 2, 3\}$. In this case, the quotient map identifies the 2-cells, 1-cells and 0-cells, i.e. $A \sim B$, $a \sim b$ and $v_1 \sim v_2$. As such we have the chain complex,

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{d_3} \mathbb{Z} \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

In this case $\operatorname{Im} d_3 = 2A$ (by the same logic as before), so $H_3(Y) \cong \mathbb{Z}$. On the other hand, $d_2 = 0$ since the identification $A \sim B$ means that $\partial A = \partial B = (a + b) - (a - b) = 0$ so $H_2(Y) \cong \mathbb{Z}_2$. As before, $d_1 = 0$ so that $H_1(Y) \cong \mathbb{Z}$. The space is again path-connected so we have:

$$H_k(Y) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 1, 3 \\ \mathbb{Z}_2 & \text{if } k = 2 \end{cases}$$

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Problem. *A map $f : S^n \to S^n$ satisfying $f(x) = f(-x), \forall x$ is called an* ***even map***. *Show that an even map $S^n \to S^n$ must have even degree and that the degree must in fact be zero when n is even. When n is odd show that $\exists$ maps of any given even degree.*

As per the hint, if f is even, then $\tilde{f} = q \circ f$, where $q : S^n \twoheadrightarrow P^n$ is the quotient map, is well-defined. In particular, since f commutes with q in the sense that $\iota \circ q \circ f = f$ where ι is the inclusion $\iota : P^n \hookrightarrow S^n$, f factors through the combination $S^n \to \mathbb{R}P^n \to S^n$. Now since $H_n(\mathbb{R}\mathrm{P}^n) = \mathbb{Z} \iff n$ is odd, the induced map on homology even gives,

$$H_n(S^n) \cong \mathbb{Z} \xrightarrow{q_* f_*} H_n(\mathbb{R}\mathrm{P}^n) \xrightarrow{\iota_*} H_n(S^n)$$

If n is even, then an even map must have degree zero since the middle term would be 0.

Now let's consider the case where n is odd. From the CW structure on $\mathbb{R}\mathrm{P}^n$ with a 1-cell in each dimension $0 \leq k \leq n$, it is clear that the pair $(\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^{n-1})$ is good pair since the n cell can retract to $\mathbb{R}\mathrm{P}^{n-1}$. Hence the long exact sequence for relative homology and proposition 2.1. gives,

$$\cdots \longrightarrow \overbrace{H_n(\mathbb{R}\mathrm{P}^{n-1})}^{0} \longrightarrow \overbrace{H_n(\mathbb{R}\mathrm{P}^n)}^{\mathbb{Z}} \longrightarrow H_n(\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^{n-1}) \cong H_n(\mathbb{R}\mathrm{P}^n / \mathbb{R}\mathrm{P}^{n-1}) \longrightarrow \overbrace{H_{n-1}(\mathbb{R}\mathrm{P}^{n-1})}^{0} \longrightarrow \cdots$$

where the last term in the sequence vanishes since $n-1$ is even. Hence $H_n(\mathbb{RP}^n) \cong H_n(\mathbb{RP}^n / \mathbb{RP}^{n-1})$. Now since $\mathbb{RP}^n / \mathbb{RP}^{n-1} \cong S^n$ (we are contracting the $n-1$-skeleton to a point). When n is odd, we know $H_n(\mathbb{RP}^n) = \mathbb{Z}$ and $H_n(S^n) = C_n(S^n) = \mathbb{Z}$ (where C_n is the n^{th} cellular chain group). But the quotient map $\mathbb{RP}^n \to \mathbb{RP}^n / \mathbb{RP}^{n-1} = S^n$ sends the generator of $C_n(\mathbb{RP}^n)$ to the generator of $C_n(S^n)$, so the quotient map in fact induces an isomorphism on the homology groups. As a result, given a map $p : S^n \to \mathbb{RP}^n$, the map $p_* : H_n(S^n) \to H_n(\mathbb{RP}^n)$ will have $p(1) = k$; define $\deg(p) = k$.

Let $g : S^n \to \mathbb{RP}^n$ be the quotient map defined above; it is claimed that $\deg(g) = 2$. To see this, let $\overline{x} \in \mathbb{RP}^n$, so $g^{-1}(\overline{x}) = \{x, -x\}$. Since g restricted to a neighborhood of x and $-x$ is a homeomorphism, the local degrees around x and $-x$ are both 1, so the total degree, the sum of the local degrees, is $\deg(g) = 2$. Hence $g_*(1) = 2$, so g_* is the doubling map. Now if $f : S^n \to S^n$ is an even map, then $f_* = h_* g_*$. Note that $f_*(1) = h_*(g_*(1)) = h_*(2)$, so $f_*(1) = 2k$ for some k, so $\deg(f) = 2k$, which is even.

Now we will show that when n is odd, there exists an even map $f : S^n \to S^n$ of any given even degree. We know $\deg(f) = 2k$, where $h_*(1) = k$. We need for a given $k \in \mathbb{Z}$, that $\exists$ an $h : \mathbb{RP}^n \to S^n$ such that $h_*(1) = k$. Note that this is pretty much done in Example 2.31. Pick k points in $\mathbb{RP}^n$, and pick pairwise disjoint neighborhoods of these k points. Let $q : \mathbb{RP}^n \to \bigvee_k S^n$ be the quotient map obtained by identifying the complement of these neighborhoods to a single point, and let $p : \bigvee_k S^n \to S^n$ identify all the summands to a single sphere. If $h = pq$, then Example 2.31 showed that $h(1) = k$.

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Problem. *compute $H_i(\mathbb{RP}^n / \mathbb{RP}^m)$ for $m < n$ by cellular homology, using the standard CW structure on $\mathbb{RP}^n$ with $\mathbb{RP}^m$ as its m-skeleton*

As before, the standard CW structure consists of 1 k-cell for all $0 \le k \le n$. Under the quotient, all cells of dimension $k \le m$ are sent to a point. Hence we have a chain complex of the form,

$$\overbrace{\mathbb{Z} \xrightarrow{d_n} \cdots \xrightarrow{d_{m+1}} \mathbb{Z}}^{n-m} \longrightarrow \overbrace{0 \xrightarrow{d_m} \cdots \mathbb{Z} \to 0}^{m}$$

This pretty much reduces to the standard case of $H_k(\mathbb{RP}^n)$ for $k > m$. In particular, we have:

$$\ker(d_i) = \begin{cases} \mathbb{Z} & i \text{ odd} \\ 0 & i \text{ even} \end{cases}$$

and

$$\mathrm{Im}(d_i) = \begin{cases} 0 & i \text{ odd} \\ 2\,\mathbb{Z} & i \text{ even} \end{cases}$$

Hence:

$$H_i(\mathbb{RP}^n / \mathbb{RP}^m) = \begin{cases} \mathbb{Z} & i = 0, m+1 \ (m \text{ odd}), n \ (n \text{ odd}) \\ \mathbb{Z}_2 & i \text{ odd}, m+1 \le i < n \\ 0 & \text{otherwise} \end{cases}$$

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Problem. *For finite CW complexes X, Y show that $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$.*

This is a straightforward computation. Firstly, note that,

$$\chi(X)\chi(Y) = \sum_i (-1)^i b_i^X \sum_j (-1)^j b_j^Y = \sum_{i,j} (-1) = \sum_{i,j} (-1)^{i+j} b_i^X b_j^Y$$

where b_i, b_j are the associated Betti numbers. Each n-cell in $X \times Y$ is the product of an i-cell in X and an $(n-i)$-cell in Y. Thus the number of n-cells in $X \times Y$ is

$$c_n = \sum_{i+j=n} b_i^X b_j^Y .$$

As such we have the result:

$$\chi(X \times Y) = \sum_n (-1)^n c_n = \sum_n (-1)^n \left(\sum_{i+j=n} b_i^X b_j^Y \right) = \sum_{i,j} (-1)^{i+j} b_i^X b_j^Y = \chi(X)\chi(Y). \tag{5}$$

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Problem. *If a finite CW complex X is the union of subcomplexes A and B, show that $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$.*

This is pretty much immediate from inclusion-exclusion. Let b_n be the number of n-cells in X, let b_n^A be the number of n-cells in A, let b_n^B be the number of n-cells in B, and let $b_n^{A\cap B}$ be the number of n-cells in $A \cap B$. As $A \cup B = X$, every n-cell in X is either in A or B or both. To find b_n, we start by considering the term $b_n^A + b_n^B$ and then via inclusion-exclusion, one sees that we need to subtract the cells in $A \cap B$. Thus we have that $c_n = b_n^A + b_n^B - b_n^{A\cap B}$. Using this, we get

$$\begin{aligned}\chi(X) = \sum_n (-1)^n b_n &= \sum_n (-1)^n (b_n^A + b_n^B - b_n^{A\cap B}) \\ &= \sum_n (-1)^n b_n^A + \sum_n (-1)^n b_n^B - \sum_n (-1)^n b_n^{A\cap B} = \chi(A) + \chi(B) - \chi(A \cap B).\end{aligned}$$

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Problem. *Suppose we build S^2 from a finite collection of polygons by identifying edges in pairs. Show that in the resulting CW structure on S^2 the 1 skeleton cannot be either of the two graphs shown on page 157, with five and six vertices.*

With this CW structure, it is clear that we can always project $S^2 \hookrightarrow \mathbb{R}^3$ onto $\mathbb{R}^2$, so that the image of the 1-skeleton of S^2 is a graph G embedded in $\mathbb{R}^2$. Moreover, we can choose this projection in such a way that the 2-simplices of the CW structure on S^2 are in bijective correspondence with the regions enclosed by the graph G.

Let the Euler characteristic of a graph have the natural definition, i.e. $\chi(G) = v - e + f$, for v is the number of vertices in G, e is the number of edges in G, and f is the number of enclosed regions. We find that for any G that is the image of the 1-skeleton of the CW structure on S^2, then $\chi(G) = \chi(S^2) = 2$. It is clear that two graphs on page 157 do not have Euler characteristic 2 for the one on the left has $v = 5, e = 10, f = 11$, so $\chi = 6$, and the one on the right has $v = 6, e = 9, f = 12$, so $\chi = 9$.

$\therefore$ The two graphs on page 157 cannot be the 1-skeleton of a CW structure on S^2.

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Problem. *Use the Mayer-Vietoris sequence to compute the homology groups of the space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus. Do the same for the space obtained by attaching a Möbius band to $\mathbb{R}\mathrm{P}^2$ via a homeomorphism of its boundary circle to the standard $\mathbb{R}\mathrm{P}^1 \subset \mathbb{R}\mathrm{P}^2$.*

a)

Let Y be the Möbius strip. Let X be the space in question and let N be a neighborhood of the identified circle in X. First let's find a good cover: $A = \mathbb{T}^2 \cup N \simeq \mathbb{T}^2$ and $B = Y \cup N \simeq S^1$, so both A and B are open with $A \cup B = X$. This yields the Mayer-Vietoris sequence

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots$$

Plugging in $H_k(A) \cong H_k(\mathbb{T}^2), H_k(B) \cong H_k(S^1)$ gives

$$H_n(A) \cong H_n(T^2) = \begin{cases} \mathbb{Z} & n = 0,2 \\ \mathbb{Z}^2 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_n(B) \cong H_n(A \cap B) \cong H_n(S^1) = \begin{cases} \mathbb{Z} & n = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

When $n \geq 3$, the Mayer-Vietoris sequence gives so $H_n(X) = 0$ for all $n \geq 3$. Looking at the low-dimensional Mayer-Vietoris sequence for reduced homology groups gives:

$$0 \longrightarrow H_2(T^2) \oplus H_2(S^1) \longrightarrow H_2(X) \xrightarrow{\psi} H_1(S^1) \xrightarrow{\Phi} H_1(T^2) \oplus H_1(S^1) \xrightarrow{\varphi} H_1(X) \longrightarrow 0$$

Using the identification of Y and T^2, the map $\Phi : H_1(S^1) \to H_1(T^2) \oplus H_1(S^1)$ is given by $\Phi(1) = ((2,0),1)$ (the boundary circle of Y gets sent twice around one of the 1-cells of T^2), so Φ is injective and $\operatorname{Im}(\Phi) = 2\,\mathbb{Z} \oplus \mathbb{Z} = \ker(\varphi)$ (the last equality because the sequence is exact). Since Φ is injective and the sequence is exact, we know ψ is the zero map, so we get the exact sequence

$$0 \to H_2(T^2) \oplus H_2(S^1) \to H_2(X) \to 0$$

As a result, $H_2(X) \cong H_2(T^2) \oplus H_2(S^1) = \mathbb{Z}$. Since the Mayer-Vietoris sequence above is exact, we see that φ is surjective, so

$$H_1(X) \cong (H_1(T^2) \oplus H_1(S^1))/\ker(\varphi) = (H_1(T^2) \oplus H_1(S^1))/\operatorname{Im}(\Phi) = \mathbb{Z}^3/(2\,\mathbb{Z} \oplus \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2$$

Adding in the fact that X is path-connected, we have:

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0,2 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

b)

Let X be the space in question, and let Y be the Möbius band in X. Let N be a neighborhood of the identified circle in X, let $A = \mathbb{R}\mathrm{P}^2 \cup N$ and let $B = Y \cup N$, so A, B are open in X and $A \cup B = X$, so we get the Mayer-Vietoris sequence

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots$$

Clearly $A \simeq \mathbb{R}\mathrm{P}^2$, $B \sim Y \sim S^1$, and $A \cap B \sim \mathbb{R}\mathrm{P}^1 \sim S^1$, so

$$H_n(A) \cong H_n(\mathbb{R}\mathrm{P}^n) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}_2 & n = 1 \\ 0 & \text{otherwise} \end{cases} \qquad H_n(B) \cong H_n(A \cap B) \cong H_n(S^1) = \begin{cases} \mathbb{Z} & n = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

As before if $n \geq 3$, $H_n(X) = 0$ for all $n \geq 3$ and X is path-connected. Now for dimensions $1, 2$:l Mayer-Vietoris sequence for reduced homology groups gives the following:

$$0 \longrightarrow H_2(X) \xrightarrow{\psi} H_1(S^1) \xrightarrow{\Phi} H_1(\mathbb{R}\mathrm{P}^2) \oplus H_1(S^1) \xrightarrow{\varphi} H_1(X) \longrightarrow 0$$

Hence ψ is injective and φ is surjective. The identification of Y and $\mathbb{R}\mathrm{P}^2$ gives a map $\Phi : H_1(S^1) \to H_1(\mathbb{R}\mathrm{P}^2) \oplus H_1(S^1)$ is defined by $\Phi(1) = (0,1)$ (the boundary circle of Y gets sent twice around $\mathbb{R}\mathrm{P}^1$, which becomes a 0 when passing to the homology group $\mathbb{Z}_2$). Thus Φ is injective, so $\mathrm{Im}(\psi) = \ker(\Phi) = 0$. ψ is then a trivial map; the only way this is possible is if $H_2(X) = 0$. This gives us a short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}_2 \oplus \mathbb{Z} \to H_1(X) \to 0$$

From the properties of short exact sequences, we thus know that $H_1(X) \cong (\mathbb{Z}_2 \oplus \mathbb{Z})/\mathbb{Z} = \mathbb{Z}_2$. Summary:

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}_2 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

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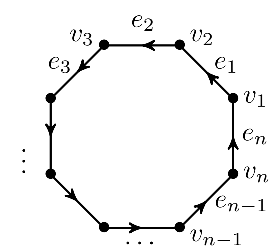
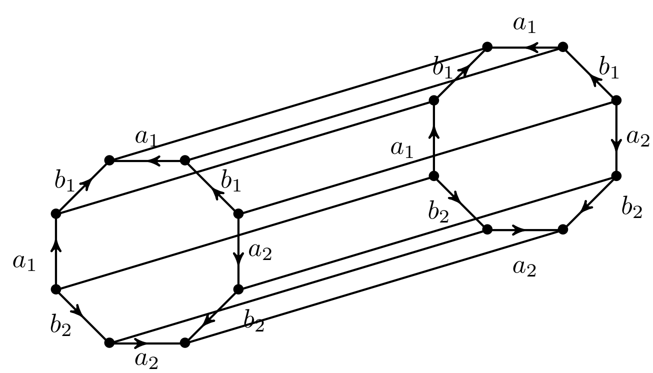
Problem. *The surface M_g of genus g, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact region R. Two copies of R, glued together by the identity map between their boundary surfaces M_g, form a closed 3-manifold X. Compute the homology groups of X via the Mayer-Vietoris sequence for this decomposition of X into two copies of R. Also compute the relative groups $H_i(R, M_g)$.*

Recall that we can draw the 1-skeleton of a surface of genus g as a $4g$-gon:



Let's consider the case $g = 2$, since we can get the higher genuses inductively. We can draw the 1-skeleton of the space X_2 as an octogonal prism:



This has a fairly straightforward cell structure, with two 2-cells $\{A, B\}$ for the edge-labelled faces, a 3-cell that connects the two faces (which become equal under the quotient) and 8 edges and 8 vertices (this is due to the quotient of the boundaries). Hence our chain complex is of the form,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_3=0} \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^8 \xrightarrow{d_1} \mathbb{Z}^8 \xrightarrow{d_0} 0$$

Under the quotient, the two 2-cells are equal so $d_3 = 0$ and $H_3(X_2; \mathbb{Z}) = \mathbb{Z}$. Now the attaching map for each 2-cell is $\prod_i [a_i, b_i]$, but this is already trivial in $\mathbb{Z}^8$ so $\mathrm{Im}\, d_2 = \prod_i [a_i, b_i]$. Note that $\ker d_1 = \prod_i [a_i, b_i]$ from the standard boundary map for M_g. Hence $H_2(X_2; \mathbb{Z}) = H_1(X_2; \mathbb{Z}) = 0$. The space is path-connected so $H_0(X_2; \mathbb{Z}) = \mathbb{Z}$. This trivially generalizes to the genus g case, X_g:

$$H_k(X_g) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 3 \\ 0 & \text{else} \end{cases}$$

We can also get this from the Mayer-Vietoris Sequence. Let $A = B = R, A \cap B = M_g$. Note that $\tilde{H}_k(R) = 0$ unless $k = 1$ when $\tilde{H}_1(R) = \tilde{H}_1(\vee_g S^1) = \mathbb{Z}^g$. Then the reduced Mayer-Vietoris sequence is:

$$\cdots \longrightarrow \tilde{H}_k(A \cap B) \cong \tilde{H}_k(M_g) \longrightarrow \tilde{H}_k(A) \oplus \tilde{H}_k(B) \cong \tilde{H}_k(R) \oplus \tilde{H}_k(R) \longrightarrow \tilde{H}_k(X) \longrightarrow \cdots$$

This is a 3-manifold, so higher groups vanish. Top cohomology reduces to the exact sequence,

$$0 \to \tilde{H}_3(X) \to \tilde{H}_2(A \cap B) \cong \mathbb{Z} \to 0$$

The other two sequences end up giving trivial groups since we find that $\tilde{H}_2(X) \cong \mathbb{Z}^{2g} / \mathbb{Z}^{2g}$ and $\tilde{H}_1(X) = 0$.

Since (R, M_g) is a good pair, $\tilde{H}_\bullet(R/M_g) \cong H_\bullet(R, M_g)$. Now it is claimed that $R/M_g \cong S^2 \vee S^1$. This is easily seen from the octogon drawing, since the quotient sends to 2-cells to a point. This gives the "earring" shape of Example 0.8, so:

$$H_n(R, M_g) \cong H_n(S^2 \vee S^1) = \begin{cases} \mathbb{Z} & n \in \{0, 1, 2\} \\ 0 & \text{Otherwise} \end{cases}$$

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Problem. *Use the Mayer-Vietoris sequence to show there are isomorphisms $\widetilde{H}_n(X \vee Y) \cong \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y)$ if the basepoints of X and Y that are identified in $X \vee Y$ are deformation retracts of neighborhoods $U \subset X$ and $V \subset Y$.*

Let x_0 be the basepoint of $X \vee Y$, with "good" neighborhoods of x_0 $U \subseteq X$ and $V \subseteq Y$ so that $X \vee Y = (X \cup V) \cup (Y \cup U)$. This gives the Mayer-Vietoris sequence

$$\cdots \to \widetilde{H}_n((X \cup V) \cap (Y \cup U)) \to \widetilde{H}_n(X \cup V) \oplus \widetilde{H}_n(Y \cup U) \to \widetilde{H}_n(X \vee Y) \to \widetilde{H}_{n-1}((X \cup V) \cap (Y \cup U)) \to \cdots$$

Since U and V deformation retract onto x_0, $X \cup V \simeq X$ and $Y \cup U \simeq Y$, so $\widetilde{H}_n(X \cup V) \cong \widetilde{H}_n(X)$ and $\widetilde{H}_n(Y \cup U) \simeq \widetilde{H}_n(Y)$. Note that $(X \cup V) \cap (Y \cup U) = (U \cup V)$, giving us the following exact sequence:

$$\cdots \to \widetilde{H}_n(U \cup V) \to \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y) \to \widetilde{H}_n(X \vee Y) \to \widetilde{H}_{n-1}(U \cup V) \to \cdots$$

By choice of "good" neighborhoods of x_0, $U \cup V$ is contractible, so $\widetilde{H}_n(U \cup V) = 0$ for all n. Hence

$$0 \to \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y) \to \widetilde{H}_n(X \vee Y) \to 0$$

$\therefore \widetilde{H}_n(X \vee Y) \cong \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y)$.

32

Problem. *For SX the suspension of X, show by a Mayer-Vietoris sequence that there are isomorphisms $\widetilde{H}_n(SX) \cong \widetilde{H}_{n-1}(X)$ for all n.*

Let a and b be the two 0-cells of SX and define $A = SX \setminus \{a\}$ and $B = SX \setminus \{b\}$. By construction, we have $A \cap B \simeq X$ and $A, B \simeq CX$ (i.e. contact the punctured cone to the base X). Note that $A \cup B = X$, so we can use the Mayer-Vietoris sequence

$$\cdots \to \widetilde{H}_n(A \cap B) \to \widetilde{H}_n(A) \oplus \widetilde{H}_n(B) \to \widetilde{H}_n(SX) \to \widetilde{H}_{n-1}(A \cap B) \to \cdots$$

Since CX is contractible, $\widetilde{H}_n(A) = \widetilde{H}_n(B) = 0$, so $\widetilde{H}_n(A) \oplus \widetilde{H}_n(B) = 0$; this gives the exact sequence

$$0 \to \widetilde{H}_n(SX) \to \widetilde{H}_{n-1}(X) \to 0$$

$\therefore \widetilde{H}_n(SX) \cong \widetilde{H}_{n-1}(X)$.

33

Problem. *Suppose the space X is the union of open sets $A_1, \dots, A_n$ such that each intersection $A_{i_1} \cap \dots \cap A_{i_k}, i_j \in \{1, \dots, n\}, i_j \neq i_k \iff j \neq k$ is either empty or has trivial reduced homology groups. Show that $\tilde{H}_i(X) = 0, \forall i \geq n-1$ and give an example showing this inequality is best possible for each n*

Suppose that $X_k = A_1 \cup \dots \cup A_k$ and $Y_k = A_k \cap \dots \cap A_n$. By construction we have $X_n = X$ and $Y_1 = \bigcap_{i=1}^n A_n$. Using induction we will show that $\forall k, 1 \leq k \leq n$, then

$$\widetilde{H}_i(X_k \cap Y_{k+1}) = 0$$

$\forall i \geq k-1$. Notice that when $k = n$, $X_k \cap Y_{k+1} = X$, so in particular this shows that $\tilde{H}_i(X) = 0$ for all $i \geq n-1$.

The base case is $k = 1$ is trivial by assumption.

Now we prove the inductive step.

$$\begin{aligned} X_k \cap Y_{k+1} &= (A_1 \cap Y_{k+1}) \cup \dots \cup (A_{k-1} \cap Y_{k+1}) \cup (A_k \cap Y_{k+1}) \\ &= (X_{k-1} \cap Y_{k+1}) \cup Y_k \end{aligned}$$

By induction, $\widetilde{H}_i(X_{k-1} \cap Y_{k+1}) = 0$ for all $i \geq k-2$. We have the following Mayer-Vietoris sequence:

$$\widetilde{H}_i((X_{k-1} \cap Y_{k+1}) \cap Y_k) \to \widetilde{H}_i(X_{k-1} \cap Y_{k+1}) \oplus \widetilde{H}_i(Y_k) \to \widetilde{H}_i(X_k \cap Y_{k+1}) \to \widetilde{H}_{i-1}((X_{k-1} \cap Y_{k+1}) \cap Y_k)$$

Observe that $(X_{k-1} \cap Y_{k+1}) \cap Y_k = X_{k-1} \cap Y_k$, and by induction $\widetilde{H}_i(X_{k-1} \cap Y_k) = 0$ for all $i \geq k-2$. Also, we know that $\widetilde{H}_i(Y_k) = 0$ for all k. We thus have the following exact sequence:

$$\widetilde{H}_i(X_{k-1} \cap Y_{k+1}) \to \widetilde{H}_i(X_k \cap Y_{k+1}) \to \widetilde{H}_{i-1}(X_{k-1} \cap Y_k)$$

By induction, both the left and right terms are zero for all $i \geq k-1$, and thus $\widetilde{H}_i(X_k \cap Y_{k+1}) = 0$ for all $i \geq k-1$. In particular, when $k = n$, we have $X_k \cap Y_{k+1} = X_n = X$, so $\widetilde{H}_i(X) = 0$ for all $i \geq n-1$.

This is the best possible situation. To see this, first notice that the smallest n we have to look at is $n = 3$. Given an $n \geq 3$ consider $X = S^{n-2}$. It is easy to see that we can decompose S^{n-2} into n open sets such that he intersection of any number of these open sets is either empty or has trivial reduced homology groups; such a **acyclic** cover is often used in sheaf cohomology. This also holds arbitrary n. However it is clear that $\widetilde{H}_{n-2}(S^{n-2}) = \mathbb{Z}$. This means that $\widetilde{H}_{n-1}(X) = 0$ so that this is the best case scenario.

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Problem. *Use the Mayer-Vietoris Sequence to show that a nonorientable closed surface X (or more generally a finite simplicial complex X for which $H_1(X)$ contains torsion) cannot be embedded a subspace of $\mathbb{R}^3$ in such a way as to have a neighborhood homeomorphic to the mapping cylinder of some map from a closed orientable surface S to X.*

Suppose that $H_1(X)$ has torsion and that $\iota : X \to \mathbb{R}^3$ is a topological embedding such that $\exists N \subset \mathbb{R}^3$ with $N \cong M$, where M is the mapping cylinder of a map $f : S \to X$. More specifically we define $M := S \times I / \sim$ where $\sim$ is the relation $(s, 1) \in S \times I \sim f(s) \in X$. By assumption, M can be embedded in $\mathbb{R}^3$. Note that this implies that N retracts onto X so that the splitting lemma gives $H_n(N) \cong H_n(X) \oplus H_n(N, X)$. Now let $\iota(X) = \tilde{X} \subset \mathbb{R}^3$ and let $A = \mathbb{R}^3 - \tilde{X}, B = N \subset \mathbb{R}^3$ so that $A \cap B = N \setminus \tilde{X}$. Now since $N \cong M$ and $\tilde{X} \cong q(S \times \{1\}) \subset M$, where $q : S \times I \to M$ is the quotient map, this means that $A \cap B \cong M - q(S \times \{1\}) \cong S \times [0, 1) \simeq S$ so that the classification of closed, orientable surfaces says that $S \cong M_g$ for some $g > 0$ and,

$$\tilde{H}_n(A \cap B) \cong \tilde{H}_n(S) = \begin{cases} \mathbb{Z}^{2g} & n = 1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Hence the (reduced) Mayer-Vietoris Sequence[1] for $(A, B, \mathbb{R}^3)$ is:

$$\begin{tikzcd}
\cdots \arrow[r, "k_* - l_*"] & \tilde{H}_{n+1}(\mathbb{R}^3) \arrow[r, "\partial"] \arrow[d, "\cong"] & \tilde{H}_n(A \cap B) \arrow[r, "(i_*, j_*)"] \arrow[d, "\cong"] & \tilde{H}_n(A) \oplus \tilde{H}_n(B) \arrow[r, "k_* - l_*"] \arrow[d, "\cong"] & \tilde{H}_n(\mathbb{R}^3) \arrow[r, "\partial"] \arrow[d, "\cong"] & \cdots \\
& 0 \arrow[r, "0"] & \tilde{H}_n(S) \arrow[r, "(i_*, j_*)"] & \tilde{H}_n(\mathbb{R}^3 - \tilde{X}) \oplus \tilde{H}_n(X) \oplus \tilde{H}_n(N, X) \arrow[r, "k_* - l_*"] & 0 &
\end{tikzcd}$$

[1] Valid since $A \cap B \neq \emptyset$

For the case of $n = 1$ this reduces to the short exact sequence,

$$0 \to \mathbb{Z}^{2g} \to \tilde{H}_1(\mathbb{R}^3 - \tilde{X}) \oplus \tilde{H}_1(X) \oplus \tilde{H}_1(N, X) \to 0 \tag{7}$$

which implies that $\mathbb{Z}^{2g} \cong \tilde{H}_1(\mathbb{R}^3 - \tilde{X}) \oplus \tilde{H}_1(X) \oplus \tilde{H}_1(N, X)$. But by hypothesis, $\tilde{H}_1(X)$ has torsion, giving a contradiction! $\therefore X$ cannot be embedded in $\mathbb{R}^3$