Math 6510 Homework 11

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§2.2 Problems

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Problem. *From the long exact sequence of homology groups associted to the short exact sequence of chain complexes* $0 \longrightarrow C_i(X) \xrightarrow{n} C_i(X) \longrightarrow C_i(X;\mathbb{Z}_n) \longrightarrow 0$*, deduce immediately that there are short exact sequences,*

$$0 \longrightarrow H_i(X)/nH_i(X) \longrightarrow H_i(X;\mathbb{Z}_n) \longrightarrow n\text{-}Torsion(H_{i-1}(X)) \longrightarrow 0$$

where n-Torsion(G) is the kernel of the map $g \mapsto ng$*. Use this to show that* $\widetilde{H}_i(X;\mathbb{Z}_p) = 0, \forall i$ *and for all primes p iff* $\widetilde{H}_i(X)$ *is a vector space over* $\mathbb{Q}$ *for all i*

The long exact sequence of interest is,

$$\begin{array}{ccccccc} \cdots \longrightarrow H_n(X) & \xrightarrow{\widetilde{n}} & H_n(X) & \xrightarrow{f_n} & H_n(X;\mathbb{Z}_m) & & \\ & & & & \Big\downarrow \partial & & \\ & & & & H_{n-1}(X) & \xrightarrow{f_{n-1}} & H_{n-1}(X) \longrightarrow \cdots \end{array}$$

where $\widetilde{n}$ is the map n after descending under the quotient. Now note that $\operatorname{Im}\widetilde{n} = \ker f$, which means that $\ker f = nH_i(X)$. The map f is surjective if $H_n(X)$ is integral homology.[1] so that by the first isomorphism theorem it induces a map $\tilde{f} : H_n(X)/\ker f = H_i(X)/nH_i(X) \xrightarrow{\cong} H_n(X;\mathbb{Z}_m)$. In the case the $H_i(X)$ has rational or real coefficients, the map $\tilde{f}$ is zero so that $H_i(X)/nH_i(X) = 0$. In the general case, we still get an injective map $\tilde{f}$ since $H_i(X)/nH_i(X)$ is the set of homologous chains in X that are not multiples of n, thus respecting the coefficients of $H_n(X;\mathbb{Z}_n)$. Now since $\operatorname{Im}\partial = \ker f_{n-1} = \text{n-Torsion}(H_{i-1}(X))$ we can form the sequence

$$0 \longrightarrow H_i(X)/nH_i(X) \xrightarrow{\tilde{f}} H_i(X;\mathbb{Z}_m) \xrightarrow{\partial} \text{n-Torsion}(H_{i-1}(X)) \xrightarrow{f_{n-1}} 0$$

Exactness of the first map implies that $\tilde{f}$ is injective, while exactness of the second map comes from the fact that $\partial f = 0$.

($\Longrightarrow$) Suppose that $\tilde{H}_i(X;\mathbb{Z}_p) = 0, \forall i$ and for all primes p. Let's first define a map $p : H_i(X;\mathbb{Z}_m) \to H_i(X)/nH_i(X)$ such that $p\tilde{f} = \mathbb{1} : H_i(X)/nH_i(X)$. In particular, if $\{\sigma_n\}$ are the generators of $H_i(X;\mathbb{Z}_m)$, since $\tilde{f}$ is an injective homomorphism, there is a subset $A \subset \{\sigma_n\}$ such that A is the images of the generators of $H_i(X)/nH_i(X)$. Hence if $[\sum_n c_n\sigma_n]$ is a chain in $H_i(X;\mathbb{Z}_m)$, then $p[\sum_n c_n\sigma_n] = \sum_{\{n:\sigma_n\in A\}} c_n\tilde{\sigma}_n$ where $\tilde{\sigma}_n$ are the generators of $H_i(X)/nH_i(X)$. By construction, we have $p\tilde{f} = \mathbb{1}$ so that by the Splitting Lemma we have $H_i(X;\mathbb{Z}_m) = H_i(X)/nH_i(X) \oplus \text{n-Torsion}(H_{i-1}(X))$. Since $\tilde{H}_i(X;\mathbb{Z}_p) = 0, \forall i, p$, this implies that $H_i(X)$ is p-torsion-less for all i and that $H_i(X)$ is infinitely-generated since $H_i(X) = pH_i(X)\forall p, i$. Note that this second fact plus unique factorization shows that $H_i(X)$ is a $\mathbb{Z}$-module and since it is a free $\mathbb{Z}$-action (due to the lack of torsion), $H_i(X)$ is a $\mathbb{Z}$-vector space. Since this means that $H_i(X) = nH_i(X)$ and any $\mathbb{Z}$-vector space is isomorphic to $\mathbb{Z}^n$ for some n, this implies that $H_i(X) \ncong \mathbb{Z}^n$ for any n. Combining all of thes fact we see that $H_i(X)$ must be free, abelian, torsion-free and infinitely-generated so $H_i(X)$ must at least be a $\mathbb{Q}$-vector space.[2]

($\Longleftarrow$) This is immediate from the splitting, since $\mathbb{Q}$ has no torsion and $H_i(X;\mathbb{Q})/nH_i(X;\mathbb{Q}) = 0$.

[1] I think this is true for any finitely-generated, Abelian group G, with $H_\bullet(X) = H_\bullet(X;G)$ as long as there exists a surjective group homomorphism $G \to \mathbb{Z}_m$

[2] There's no reason that we can't extend this to any field of characteristic zero, right? If there is an analogue of sheaf cohomology but for homology, perhaps we can choose the local ring of units of the sheaf of holomorphic functions $\mathcal{O}_X$.

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Problem. *For X a finite CW complex and F a field show that the Euler Characteristic $\chi(X)$ can also be computed by the formula $\chi(X) = \sum_n (-1)^n \dim H_n(X; F)$, the alternating sum of the dimensions of the vector spaces $H_n(X; F)$.*

Let $C_k := H_k(X^n, X^{n+1})$ be the Cellular Chain Groups. Then if $c_k = |C_k|$, the Euler Characteristic is defined as $\chi(X) = \sum_k (-1)^k c_k$. The Cellular Chain Complex is,

$$0 \longrightarrow C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

If we let B_k be the set of k-boundaries and Z_k be the set of k-cycles, then we have the short exact sequences of vector spaces,

$$0 \longrightarrow Z_k \longrightarrow C_k \xrightarrow{d_k} B_{k-1} \longrightarrow 0$$
$$0 \longrightarrow B_k \longrightarrow Z_k \longrightarrow H_k \longrightarrow 0$$

where $H_k = Z_k/B_k$. In particular, this gives $c_k = \dim C_k = \dim Z_k + \dim B_k = \dim H_k + \dim B_{k-1}$ so that

$$\chi(X) = \sum_k (-1)^k \dim Z_k + \sum_k (-1)^k \dim B_k + \sum_k (-1)^k \dim B_{k-1} = \sum_k (-1)^k \dim H_k$$

where the last equality comes from the fact that the B_k, B_{k-1} sums form a telescoping series. In particular since $H_k = H_k(X; F)$, this is the desired result.

§1.3 Problems

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Problem. *Show that if $p_1 : \tilde{X}_1 \to X_1$ and $p_2 : \tilde{X}_2 \to X_2$ are covering spaces, so is their product $p_1 \times p_2 : \tilde{X}_1 \times \tilde{X}_2 \to X_1 \times X_2$*

By definition, this implies that $\exists$ covers $\{U_i^1\}, \{U_j^2\}$ of $\tilde{X}_1, \tilde{X}_2$ such that $\forall x_1 \in X_1, x_2 \in X_2$ there exists $U_i^1 \ni x_1, U_i^2 \ni x_2$ such that $p_1^{-1}(U_i^1) \cong \sqcup_\alpha \tilde{U}_\alpha^1, p_2^{-1}(U_i^2) \cong \sqcup_\beta \tilde{V}_\beta^2$ with $\tilde{U}_\alpha^1 \cong U_i^1, \tilde{V}_\beta^2 \cong U_i^2, \forall \alpha, \beta, i$. Finally note tht if $(x, y) \in X_1 \times X_2$, then $(p_1 \times p_2)^{-1}(x, y) = \{(u, v) \mid u \in p_1^{-1}(x), v \in p_2^{-1}(y)\}$.

As $\{U_i^1\}$ covers X_1 and $\{U_i^2\}$ covers X_2, then $\{U_i^1 \times U_i^2\}$ is an open cover of $X_1 \times X_2$. Let $(x, y) \in X_1 \times X_2$, so there exists a $U \in \{U_i^1\}$ containing x and a $U' \in \{U_i^2\}$ containing y with $p_1^{-1}(U) = \sqcup_\alpha \tilde{U}_\alpha^1$ and $p_2^{-1}(U') = \sqcup_\beta \tilde{V}_\beta^2$. Thus $U \times U' \in \{U_i^1 \times U_i^2\}$ is a neighborhood of (x, y) with

$$(p_1 \times p_2)^{-1}(U \times U') = \sqcup_{\alpha,\beta} \tilde{U}_\alpha^1 \tilde{V}_\beta^2$$

This is a disjoint union of open sets in $\widetilde{X}_1 \times \widetilde{X}_2$, since $\tilde{U}_\alpha \cap \tilde{U}_{\alpha'} = \emptyset, \tilde{V}_\beta \cap \tilde{V}_\beta = \emptyset$ for $\alpha \neq \alpha', \beta \neq \beta'$, and $\tilde{U}_\alpha \times \tilde{V}_\beta \cong U \times U'$. Thus $(\widetilde{X}_1 \times \widetilde{X}_2, p_1 \times p_2)$ is a covering space of $X_1 \times X_2$.

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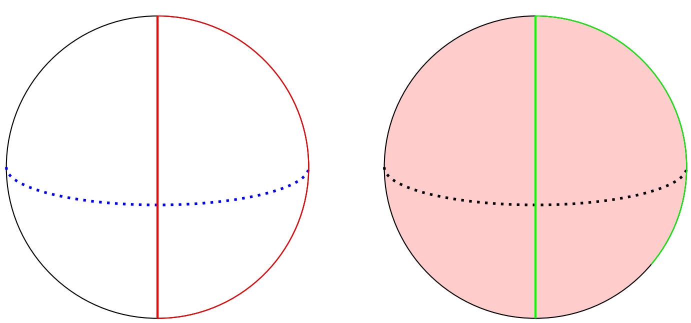
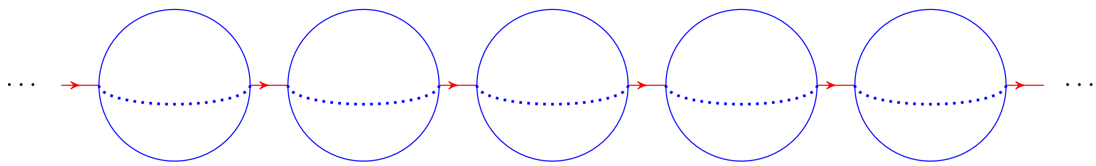
Problem. *Construct a simply-connected covering space of the space $X \subset \mathbb{R}^3$ that is the union of a sphere and a diameter. Do the same when X is the union of a sphere and a circle intersecting it in two points*

Let's first describe some of the geometric intuition attached to this space. Pictorally, we have,



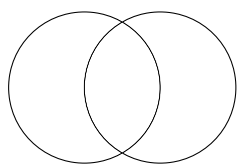
Now since S^2 is simply connected, any non-trivial loop must pass through the diameter. For example, the path in red (on the left sphere) is a loop that is not homotopic to a trivial loop. As such, it is easy to conjecture that the fundamental group of this space is $\mathbb{Z}$; this can be formalized using van Kampen's theorem by considering the path-connected open sets $A_1, A_2 \subset X$ with $A_1 \cap A_2$ path-connected. For example, suppose that A_1 is equal to the sphere minus the diameter so that $A_1 \cong S^2 - \{N, S\} \cong S^1 \times \mathbb{R}$ and let A_2 be the union of a half-open arc on S^2 containing the endpoint of the diameter and the diameter. In the above picture, A_1 is shaded in red while A_2 is in green. The intersection $A_1 \cap A_2$ is an open arc so $A_1 \cap A_2$ is path-connected and $\pi_1(A_1 \cap A_2, x_0) = 0$ for any $x_0 \in A_1 \cap A_2$. Moreover, $\pi_1(A_1) = \mathbb{Z}, \pi_1(A_2) = 0$ so that van Kampen's theorem gives $\pi_1(X) = \mathbb{Z}$.

Given these (somewhat) geometric facts, intuition dictates that one should try to construct a covering space analogous to the covering space $\mathbb{R} \to S^1$. In particular, since the generator of $\pi_1(X)$ is any loop that goes through the diameter once, we can generalize the helix construction by joining together copies of spheres linked together by closed intervals. This way, a degree k loop lifts to a path that passes through k spheres and k closed intervals. This space Y is,

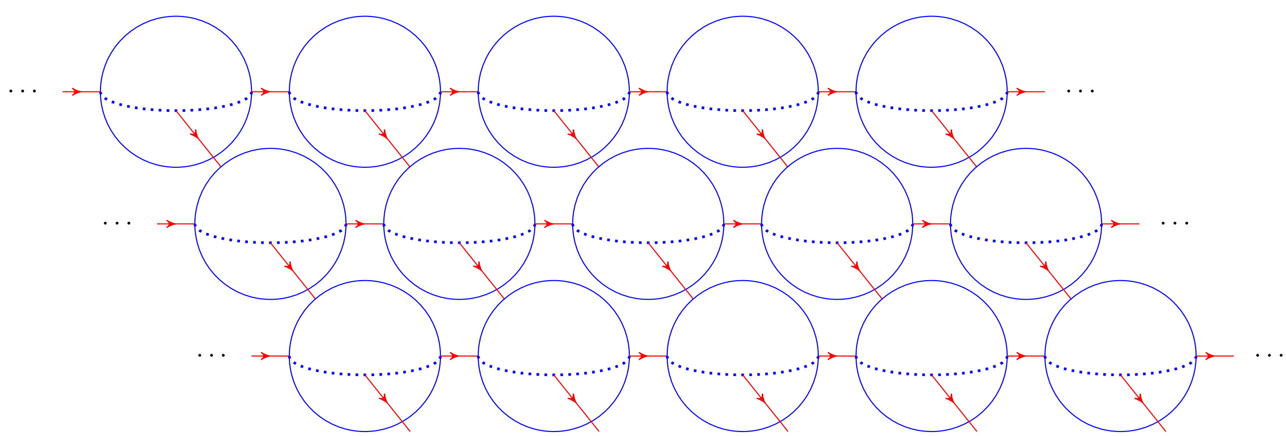


This is simply connected since it is homotopy equivalent to the wedge sum of an infinite number of 2-spheres; the fact that it is a covering space is effectively the same proof that $\mathbb{R}$ covers S^1. It is clear that anywhere on $S^2 - \{N, S\}$ or on the interior of the diameter, this defined a covering space, so we only have to consider the cover of N, S. However, this is also straightforward, for if we choose an orientation on each of the closed interval pieces, we distinguish endpoints of the intervals in Y and map them to N, S.

Let's again approach this geometrically. By choosing intersecting a plane that contains the intersecting circle and looking at the intersection of this plane with X, we will get a 1-dimensional shape that looks like,



We can use this as a 1-skeleton X^1 for X. Using van Kampen's theorem, it is clear that we can show that $\pi_1(X^1) \cong \mathbb{Z}^3$, which each generator corresponding to either a loop in one of the circles or a loop in the "intersecting" circle (actually, more like a not-so-smooth ellipse). When we attach the 2-cells to form X, an application of Proposition 1.26 tells us that $\pi_1(X) = \mathbb{Z}^2$. Given this, we can attempt to construct a lattice of spheres in the same spirit at the previous covering space, except that we are instead mimicking the covering space of $\mathbb{T}^2$. This space looks like,



This is again simply connected because it is homotopy equivalent to the wedge sum of an infinite number of copies[3] of S^2 and the argument that this is a covering space is the same as before.

[3]One thing that I'm not sure of is if the morphism π_1 deals well with infinite wedge sums since I'm not sure if the free product has any constraints on "finitely-generated." I don't think that the free product has any finitely-generated constraints, but regardless it will not affect this problem.

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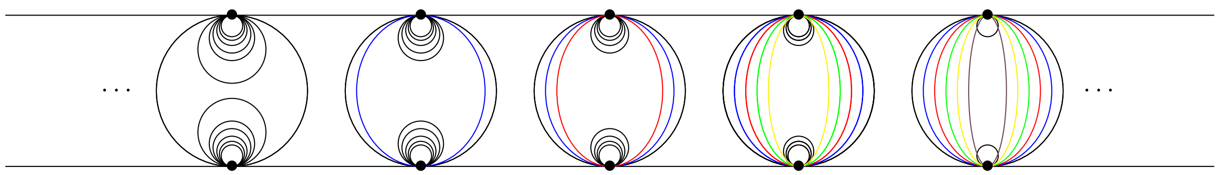
Problem. *Let X be the Hawaiian Earring and let $\tilde{X}$ be the covering space on page 79. Construct a two-sheeted covering space $Y \to \tilde{X}$ such that the composition $Y \to \tilde{X} \to X$ of the two covering spaces is not a covering space.*

Consider the following space Y,



This is a 2-sheeted covering space of the chain of Hawaiian Earrings on page 79, since any neighborhood of a vertex in $\widetilde{X}$ is covered by two copies of the same neighborhood in the above space. The key feature here is presence of loops that connect the two vertices. Hence the inverse image of the second-to-outer most loop in X in $\widetilde{X}$ is an infinite collection of loops each of which have a different inverse image in the space above. In particular, an "outer" loop in $\widetilde{X}$ as an inverse image in Y corresponding to a different integer number of loops; in the diagram above, these are the colored loops. Hence, Y does not cover X

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Problem. *Let Y be the quasi-circle, a closed subspace of $\mathbb{R}^2$ consisting of a portion of the graph $y = \sin(1/x)$, the segment $[-1, 1]$ in the y-axis and an arc connecting these two pieces. Collapsing the segment of Y in the y-axis to a point gives a quotient map $f : Y \to S^1$. Show that f does not lift to the covering space $\mathbb{R} \to S^1$, even though $\pi_1(Y) = 0$. Thus the local path-connectedness of Y is a necessary hypothesis in the lifting criterion*

Suppose that we have the quotient map $f : Y \to S^1$so that our goal is to show that there does not exist a lift $\widetilde{f} : Y \to \mathbb{R}$. Now suppose that f does lift to $\widetilde{f} : Y \to \mathbb{R}$ and let $\gamma : [0, 1] \to S^1$ be a loop that travels around S^1 once (degree 1), so in particular γ is not homotopic to the constant loop. Hence if $\widetilde{\gamma}$ is a lift of γ, then $\widetilde{\gamma}(0) \neq \widetilde{\gamma}(1)$. Now consider the map $\gamma' = f^{-1}(\gamma) : [0, 1] \to Y$. Since $\widetilde{f}$ is a lift of f, then (letting $p : \mathbb{R} \to S^1$ be the covering map), $p\widetilde{f}(\gamma) = \gamma$. However, by choice of γ (i.e. choosing γ such that its not nullhomotopic), in order to have $p\widetilde{f}(\gamma') = \gamma$, γ' would have to travel across the oscillatory part of $\sin(1/x)$ and to the segment in the y-axis, which is impossible, since Y is not locally path-connected at the limit line-segment of $\sin(1/x)$.
$\therefore \nexists$ such a lift $\widetilde{f}$

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Problem. *Show that if a path-connected, locally-path connected space X has $\pi_1(X)$ finite, then every map $X \to S^1$ is nullhomotopic*

Since $\pi_1(X)$ is finite, it is finitely generated and any map $f : X \to S^1$ induced a map $f_* : \pi_1(X) \to S^1$. If f_* is trivial, then the map is nullhomotopic so lets assume that f_* is non-trivial. In this case, at least one generator of $\pi_1(X)$ must map to a non-trivial element of $\pi_1(S^1) \cong \mathbb{Z}$. Let this generator be $g \in \pi_1(X)$ so that by finiteness, there exists an $n \in \mathbb{N}$ such that $g^n = \mathbb{1}$. Since f_* is a homomorphism, this implies that $\mathbb{1}_{\pi_1(S^1)} = f_*(\mathbb{1}_{\pi_1(X)}) = f_*(g^n) = f_*(g)^n$. But this implies that $f_*(g) \neq \mathbb{1}_{\pi_1(S^1)}$ is an element of finite order in $\pi_1(S^1)$, which cannot be true. Hence f_* must be trivial.

Now let $p : \mathbb{R} \to S^1$ be the helical covering map of S^1. From Proposition 1.33, $\exists \tilde{f} : \mathbb{R} \to X$ iff $f_*(\pi_1(X)) \subseteq p_*(\pi_1(\mathbb{R})) = 0$, which is true from the preceding paragraph. Now any lift $\tilde{f} : X \to \mathbb{R}$ is nullhomotopic by the homotopy $\tilde{f}_t(X) = (1-t)\tilde{f}(x) + tf(x_0)$, where x_0 is the distinguished basepoint. Hence $f_t = p\tilde{f}_t : X \to S^1$ is a nullhomotopy of f.

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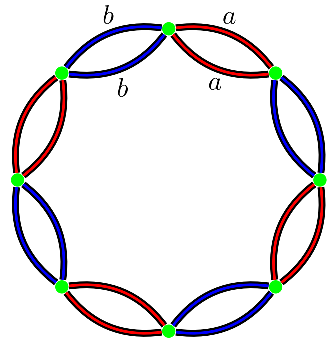
Problem. *Let a and b be the generators of $\pi_1(S^1 \vee S^1)$ corresponding to the two S^1 summands. Draw a picture of the covering space of $S^1 \vee S^1$ corresponding to the normal subgroup generated by $a^2, b^2, (ab)^4$ and prove that this covering space is indeed the correct one.*

Since a covering space of $S^1 \vee S^1$, thought of as a graph, is a graph, we first try to draw graphs with the desired relations. The relations a^2, b^2 are much easier to add into an arbitrary graph, so let's try to construct a graph such that $(ab)^4 = e$. In particular, an octogon (as a cycle) will satisfy this relation. Hence we have,



where the blue edges correspond to b and the red edges correspond to a. This is a covering space since all covers of $S^1 \vee S^1$ need to satisfy the condition that each node serves at the endpoint for an incoming and an outgoing edge for each generator a, b.

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Problem. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $\varphi(x, y) = (2x, y/2)$. This generates an action of $\mathbb{Z}$ on $X = \mathbb{R}^2 - \{0\}$. Show this action is a covering space action and compute $\pi_1(X/\mathbb{Z})$. Show that the orbit space is non-Hausdorff and describe how it is a union of four subspaces homeomorphic to $S^1 \times \mathbb{R}$, coming from the complementary components of the x-axis and the y-axis*

Firstly note that the hyperbolas $xy = c, c \in \mathbb{R}$ are invariant under the action induced by φ. Let's show that φ induces a covering space action.

Covering Space Action

The $\mathbb{Z}$-action induced by φ is $n \cdot (x, y) = (2^n x, y/2^n)$. We need to show that for each $(x, y) \in \mathbb{R}^2$, there exists a neighborhood $U \subset \mathbb{R}^2$ of (x, y) such that $g_1 \neq g_2$ implies $g_1(U) \cap g_2(U) = \emptyset$. Now consider a closed ball B of radius $r = \sup_{x', y' \in B} \sqrt{(x' - x)^2 + (y' - y)^2}$ around a point $(x, y) \in \mathbb{R}^2$. Note that $B_n := n \cdot B$ is an ellipsoid centered at $(2^n x, y/2^n)$ with axes of length $a_n = 2^{-n} r_n, b = 2^n r_n$, where $r_n = \sup_{x', y' \in B} \sqrt{2^{2n}(x' - x)^2 + 2^{-2n}(y' - y)^2}$. We want to show that there exists $r > 0$ such that $B_n \cap B_m = \emptyset$. The distance $d_{n,m}$ between the centers of the two balls B_n, B_m is defined by

$$d_{n,m}^2 = (2^n - 2^m)^2 x^2 + (2^{-n} + 2^{-m})^2 y^2$$

We want $d_{n,m} > \max\{a_n, a_m\}, d_{n,m} > \max\{b_n, b_m\}$ for all n, m. However if we choose an arbitrary $(x', y') \in \partial B$ that gives r_n, this constraint reduces to two inequalities to solve for two unknown variables (x', y'). A choice of x', y' determines r and hence such an open set U exists.
$\therefore \varphi$ induces a covering space action

X is not Hausdorff

Firstly note that the family of hyperbolas $xy = c$ give a 1-foliation of $\mathbb{R}^2$. This means that for each $(x', y') \in \mathbb{R}^2, \exists c \in \mathbb{R}$ such that $x'y' = c$. Since each hyperbola $xy = c$ is invariant under the $\mathbb{Z}$-action of φ, each hyperbola corresponds to a distinct point in the quotient $X/\mathbb{Z}$. However, every hyperbola in the family $xy = c$ intersects. To see this, suppose that $xy = c_1$ and $xy = c_2$ are hyperbolas in $\mathbb{R}^2$. Now for $n, m \in \mathbb{N}$ define $x_n = c_1/n, y_n = n$ and $x'_m = c_2/m, y'_m = m$ so that $x_n y_n = c_1, x'_m y'_m = c_2$. However given $\epsilon > 0$, it is clear that for some $N, M \in \mathbb{N}$, we have $|x_n - x'_n| < \epsilon$ so that as $n, m \uparrow \infty$, $x_n \to x'_n$. The same holds if we swap the definitions of x_n, y_n. Hence every hyperbola intersects and thus under the quotient, no two points can have disjoint neighborhoods since every point is on a hyperbola.

$\pi_1(X/\mathbb{Z}) = \mathbb{Z}^2$

Since $X = \mathbb{R}^2 - \{0\} \simeq S^1$, $\pi_1(X) \cong \mathbb{Z}$ and since X is path-connected and locally path-connected, we can try to use Proposition 1.40 to deduce $\pi_1(X/\mathbb{Z})$. From the Proposition we have $\pi_1(X/\mathbb{Z})/p_*(\pi_1(X)) = \pi_1(X) = \mathbb{Z}$; moreover from the injectivity of p_* and $p_*(\pi_1(X)) \subset \pi_1(X/\mathbb{Z})$, we know that $p_*(\pi_1(X)) = \mathbb{Z}$ so that $\pi_1(X/\mathbb{Z}) = \mathbb{Z}^2$

Finally, we need to describe how $X/\mathbb{Z}$ is the union of four subspaces homeomorphism to $S^1 \times \mathbb{R}$. Consider the four quadrants of $\mathbb{R}^2 - \{0\}$ and define the $\mathbb{R}$ in $S^1 \times \mathbb{R}$ to be the distance from the origin in the quadrant we are focusing on. The orbits of the action by $\mathbb{Z}$ are all contained in the hyperbolas $xy = c$ where c is in the $\mathbb{R}$-coordinate. However, since $\mathbb{Z}$ is a discrete doubling action, the entirety of each hyperbola is not in one equivalence class. In fact, it is easy to see that the action of $\mathbb{Z}$ on a single hyperbola is equivalent to the action of $\mathbb{Z}$ on $\mathbb{R}$ given by translation. As a result, the orbit space of this action on a single hyperbola is the circle S^1 (as this is the orbit space of $\mathbb{Z}$ acting on $\mathbb{R}$ by translation). Thus we get a copy of S^1 for each $c > 0$, giving us that the orbit space of $\mathbb{Z}$ acting on a single quadrant is $S^1 \times (0, \infty) \cong S^1 \times \mathbb{R}$. This holds for all four quadrants, so $X/\mathbb{Z}$ is the union of four subspaces homeomorphic to $S^1 \times \mathbb{R}$.