Math 6510 Homework 2

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Problem. *If X_0 is the path-component of a space X containing the basepoint x_0, show that the inclusion $X_0 \hookrightarrow X$ induces an isomorphism $\pi_1(X_0, x_0) \to \pi_1(X, x_0)$.*

We know that $\iota_* : \pi_1(X_0, x_0) \to \pi_1(X, x_0)$ is a group homomorphism and moreover that ι_* must be injective since any loop based at x_0 and in X_0 must be a loop in X as well. Hence we only need to show that ι_* is surjective. Suppose $[x] \in \pi_1(X, x_0)$. Since $x : I \to X$ is a loop, it is in particular a path from x_0 to x_0. This means that x needs to be a continuous map. Hence any loop based at x_0 must be contained within X_0 since the map would not be continuous if $\mathrm{Im}(x) \subset X$ intersects a different component. Thus there must be some class of loops $[x'] \in \pi_1(X_0, x_0)$ such that $\iota_*([x']) = [x]$.

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Problem. *Show that every homomorphism $\pi_1(S^1) \to \pi_1(S^1)$ can be realized as the induced homomorphism φ_* of a map $\varphi : S^1 \to S^1$.*

Suppose that $f : \pi_1(S^1) \to \pi_1(S^1)$ is a homomorphism. Since $\pi_1(S^1) \cong \mathbb{Z}$, the image of f is completely determined by $f([\omega_1])$ (or $f(1)$ if we consider f as a map $\mathbb{Z} \to \mathbb{Z}$). Since f maps loops to loops, that means that there is some class of loops $[\omega_k], k \in \mathbb{Z}$ such that $f([\omega_1]) = [\omega_k]$. However, note that the degree k map $\varphi_k : S^1 \to S^1, \theta \mapsto k\theta$ or $e^{i\theta} \mapsto e^{ik\theta}$ has an induced map that sends $[\omega_1]$ to $[\omega_k]$. To see this, let $\gamma : I \to S^1$ be the loop $\gamma(x) = e^{2\pi i x}$. Then $\varphi_k \circ \gamma(x) = e^{2\pi i k x}$ which is simply ω_k. Since we've shown that $(\varphi_k)_*([\omega_1]) = f_*([\omega_1])$, this means that $(\varphi_k)_* \equiv f_*$.

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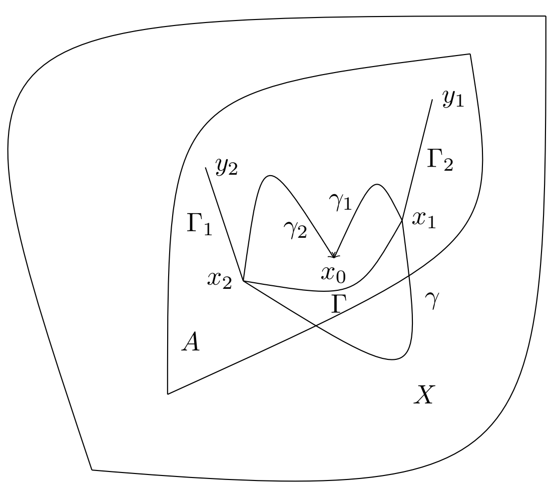
Problem. *Given a space X and a path-connected subspace A containing the basepoint x_0, show that the map $\pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $A \hookrightarrow X$ is surjective iff every path in X with endpoints in A is homotopic to a path in A*

($\Rightarrow$) Suppose that ι_* is surjective and let $x_0 \in A$ be a basepoint. This means that every loop based at $x_0 \in A \subset X$ in X is homotopic to a loop contained in A. Since A is path-connected, we know that $\pi_1(A, x_0) \cong \pi_1(A, x_1), \forall x_0, x_1 \in A$. Now suppose that $\gamma : I \to X, \gamma(0) = x_1, \gamma(1) = x_2, x_1, x_2 \in A$ is a path in X with endpoints in A. Since A is a path-connected, $\exists$ paths $\gamma_1 : I \to A, \gamma_1(0) = x_0, \gamma_1(1) = x_1$ and $\gamma_2 : I \to A, \gamma_2(0) = x_0, \gamma_2(1) = x_2$. Hence $\bar{\gamma}_1 \cdot \gamma \cdot \bar{\gamma}_2$ is a loop in X based at x_0. This is more easily seen in the following diagram:



Since ι_* is surjective, $\exists$ a loop $\Gamma : I \to A, \Gamma(0) = \Gamma(1) = x_0$ that is homotopic to $\bar{\gamma}_1 \cdot \gamma \cdot \bar{\gamma}_2$. Now denote[1] $y_1 := \Gamma\left(\frac{1}{3}\right) \in A, y_2 := \Gamma\left(\frac{2}{3}\right) \in A$. Since A is path-connected, there exists paths $\Gamma_1 : I \to A, \Gamma_1(0) = y_1, \Gamma_1(0) = x_1$ and $\Gamma_2 : I \to A, \Gamma_2(0) = y_2, \Gamma_2(0) = x_2$. Now we can construct a path $\eta : I \to A$ between x_1, x_2 as,

$$\eta(t) = \begin{cases} \bar{\Gamma}_1(3t) & t \in [0, \frac{1}{3}] \\ \Gamma(t) & t \in [\frac{1}{3}, \frac{2}{3}] \\ \Gamma_2(3t-2) & t \in [\frac{2}{3}, 1] \end{cases} \tag{1}$$

See the diagram for intuition.

($\Leftarrow$) This is straightforward: If every path in X with endpoints in A is homotopic to a path in A, then a loop (which is also a path) based at x_0 in X is homotopic to a loop based at x_0 in A. This is precisely the statement that $\pi_1(A, x_0) \to \pi_1(X, x_0)$ is surjective.

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Problem. *Show that there are no retractions $r : X \to A$ in the following cases:*

- *(a). $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1*
- *(b). $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$*
- *(c). $X = S^1 \times D^2$ with A the circle in the figure on page 39*
- *(d). $X = D^2 \vee D^2$ with A its boundary $S^1 \vee S^1$*
- *(e). X a disk with two points on its boundary identified and A its boundary $S^1 \vee S^1$*
- *(f). X the Möbius band and A its boundary circle*

(a)

Suppose that a retraction $r : \mathbb{R}^3 \to A, A \cong S^1$ existed. Then proposition 1.15 implies that $\exists$ a homomorphism $\{0\} = \pi_1(\mathbb{R}^3) \hookrightarrow \pi_1(A) = \mathbb{Z}$, a contradiction.

(b)

Suppose that $\exists$ a retraction $r : S^1 \times D^2 \to \partial(S^1 \times D^2) = S^1 \times S^1$. Since S^1, D^2 are path-connected, proposition 1.12 implies that $\pi_1(S^1 \times D^2) \cong \pi_1(S^1) \times \pi_1(D^2) \cong \mathbb{Z}, \pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$. Since r exists, this means that we have an injective homomorphism $\mathbb{Z}^2 \to \mathbb{Z}$, a contradiction.

(c)

Suppose that such a retraction $r : X \twoheadrightarrow A$ exists. Then this implies that we can contract $S^1 \times D^2$ to the non-trivial knot (It's the Torus Knot $K_{1,1}$) A. However, $\pi_1(S^1 \times D^2) \cong \pi_1(S^1) \times \pi_1(D^2) \cong \mathbb{Z}$ which implies that $S^1 \times D^2$ retracts onto the central, meridional S^1. However, A is a non-trivial knot and by the discussion on Torus Knots in section 1.2 and the introduction to chapter 1, we know that there doesn't exist an isotopy (or even a homotopy) between the unknot (the meridional circle) and the non-trivial knot in A. Since A has two links, the Wirtinger Presentation theorem guarantees that $\pi_1(A)$ has at least two generators. This implies that there is no injective homomorphism of $\pi_1(A) \to \pi_1(X)$.

(d)

Suppose that $\exists$ a retraction $r : D^2 \vee D^2 \to S^1 \vee S^1$. Since $D^2 \vee D^2$ is contractible to the join point, $\pi_1(D^2 \vee D^2) = 0$. On the other hand, $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$, so that the existence of r and proposition 1.1.5 imply that there is an injective group homomorphism $\mathbb{Z} * \mathbb{Z} \to 0$, a blatant contradiction.

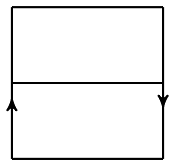
(e)

A disk with two points identified is homotopic to S^1 (see picture). If such a retraction $D^2/\sim \to S^1 \vee S^1$ existed then this means that there exists an injective group homomorphism $\mathbb{Z} * \mathbb{Z} = \pi_1(S^1 \vee S^1) \to \pi_1(D^2/\sim) = \pi_1(S^1) = \mathbb{Z}$.

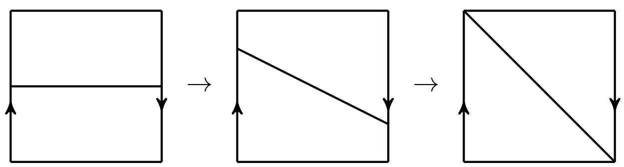
[1] The choice of $\frac{1}{3}, \frac{2}{3}$ is arbitrary; in general one can choose arbitrary points in I)

(f)

Suppose that X is the Möbius Strip and A is a component of $\partial X \cong S^1 \sqcup S^1$. Now suppose that there exists a retraction $r : X \twoheadrightarrow A$. Recall that the Möbius Strip has the following fundamental polygon:



The above line represents the "equatorial" line on the Torus. Since retractions take loops to loops, this means that the above loop would be mapped to a loop in A. This means that the above line could be send up to the top or bottom of the above polygon. However, because of the quotient, we have to continuously deform the loop in the following manner:



which is not the same as what would happen under the retraction.

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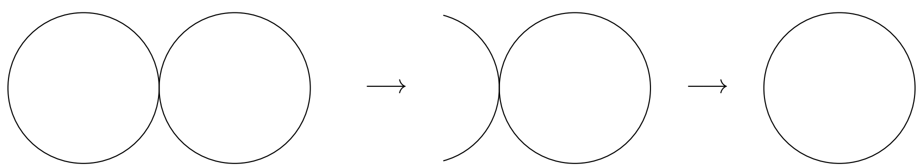
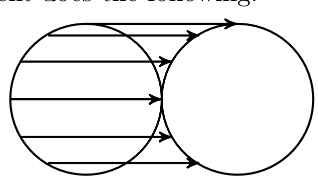
Problem. *Show that there are infinitely many retractions $S^1 \vee S^1 \to S^1$*

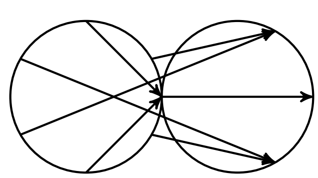
Firstly, we know of the following type of retraction, which simply contracts one circle to a point:



Now geometrically it is easy to picture (were wedge sum embedded in $\mathbb{R}^2$) "folding the circles" along their intersecting point (x_0, y_0). More formally, this is equivalent to projecting a point of one of the circles onto the other via a quotient. In particular, modding out by the equivalence relation $\theta_0 \sim \theta_1 \iff e^{i\theta_0} = e^{i\theta_1}$, where θ_0, θ_1 are the angle forms on the two circles of $S^1 \vee S^1$. Pictorically, this quotient does the following:



Now note that this quotient is constructed by looking at what is effectively the identity map: If θ_0, θ_1 are the same relative to a fixed abscissa (say, the positive x-axis), then we set θ_0, θ_1 to be equal under the quotient. Similarly, we could define a quotient $\theta_0 \sim \theta_1 \iff e^{in\theta_0} = e^{i\theta_1}$ for some $n \in \mathbb{Z}$. This is effectively a "twisting" of one circle as we project onto the second. For example, the mapping in the $n = 2$ case (i.e. the doubling map) is depicted below.

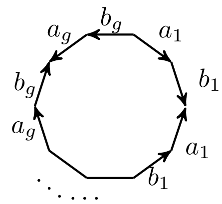


Hence the map that retracts onto the S^1 with angle form θ_1 has an infinite number of retractions that arise via a quotient map of the form $e^{in\theta}$. Note that the degree n and degree m maps are homotopic $\iff m = n$, for otherwise the induced homomorphism from the degree n map could take $[\omega_n]$ to $[\omega_m]$, a contradiction. In fact, since the set $\{e^{in\theta}\}_{n\in\mathbb{Z}}$ is a basis for $L^2(S^1, d\theta)$, we can use any measurable function $S^1 \to \mathbb{R}$ with norm 2π to define the quotient. To see this, simply consider the Fourier Decomposition of $f \in L^2(S^1, d\theta)$ as $f(\theta) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}$. Recall that Parseval's Identity says that $\sum_{n\in\mathbb{Z}} |a_n|^2 = \frac{1}{2\pi}\int_{S^1} |f(x)|^2 dx$. Hence if $\|f\|_{L^2(S^1)} = 2\pi$, we see via the decomposition that f maps S^1 into S^1. Intuitively, this means that the function f correctly measures the circumference of the unit S^1.

Extra

Problem. *Let X be the closed orientable surface of genus 2. Construct a homomorphism from the fundamental group of X onto a free abelian group of rank 4 (the direct sum of four copies of an infinite cyclic group) such that the four curves labeled a, b, c, d in the figure are sent to generators of the four infinite cyclic direct summands.*

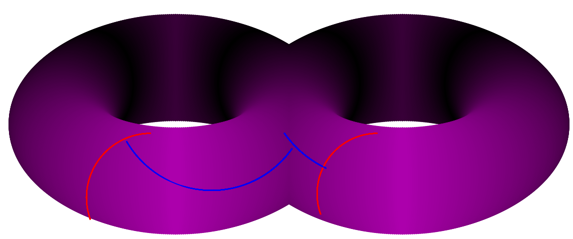
Let's first try to do this intuitively, using the fundamental polygon for a oriented, compact, connected genus g surface. After gaining some intuition, we can use van Kampen's Theorem to construct $\pi_1(\Sigma_g)$, where Σ_g is the unique (up to homeomorphism) oriented, compact, connected genus g surface. Topologically, we can construct this using a 2-dimensional CW complex. Suppose that we have $4g$ 0-cells that are connected by 1-cells (arcs) such that we have a $4g$-gon for the 1-skeleton of our CW complex. The image of this is below:



Now attach a 2-cell along the word $a_1b_1a_1^{-1}b_1^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}$. By this construction of the 1-skeleton, the word $a_1b_1a_1^{-1}b_1^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}$ is empty. Up to this point our CW complex is homeomorphic to a hemisphere. Now quotient by the relations indicated on the 1-skeleton to get the genus g surface. This is a relation that is added to the relation for the wedge sum of g copies of S^1. Intuition tells us that since the two generators of each $\pi_1(\mathbb{T}^2) \hookrightarrow \pi_1(\Sigma_g)$ commute, we should have a two figure eights at the points $\{x_i\}_{i=1}^{g-1}$ used to define the connected sum $\Sigma_g \cong \mathbb{T}^2 \# \mathbb{T}^2 \# \cdots \# \mathbb{T}^2$. As such, we can take a guess at $\pi_1(\Sigma_g)$, namely (the quotations are there because we haven't used basepoints):

$$\pi_1(\Sigma_g)\text{"} \cong \text{"}(*_{i=1}^{g}\,\mathbb{Z}^2)/(\{[a_i,b_i]=1, \forall i \leq g\}) \tag{2}$$

Now this is quite close to the actual group presentation, except that the a_i, b_i are all at different basepoints. Based on the fundamental polygon, if the basepoint x_0 is the first vertex (i.e. the vertex before the a_1, then we can construct each a_i, b_i (the two loops on the i^{th} copy of $\mathbb{T}^2 \setminus \{p\} \hookrightarrow \Sigma_g$, by using the change of basepoint homomorphism to map $x_0 = (\theta_1, \phi_1)$ (on a chart around the copy of $\mathbb{T}^2$ that x_0 is on). This is more easily seen in the following diagram, where the basepoint change homomorphism for the longitudinal loops:



As such, we can construct $\pi_1(X, x_0)$ by constructing all of the a_i in this manner. However, the commutation rule for loops in the same embedded submanifold[2] changes when we fix our basepoint. As one might guess from the fundamental polygon the **only** relation we end up getting is $a_1b_1a_1^{-1}b_1^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}$ so that we have:

$$\pi_1(\Sigma_g) \cong \langle a_1, b_1, \dots, a_g, b_g | a_1b_1a_1^{-1}b_1^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}\rangle \tag{3}$$

Hence for Σ_2, we have a simple map $\pi_1(\Sigma_g) \to \mathbb{Z}^2 \times \mathbb{Z}^2$, since $\mathbb{Z}^2 \times \mathbb{Z}^2 = \pi_1(\Sigma_g)/\{[a_i, b_j]\}$, where $[a_i, b_j]$ is the commutator subgroup $a_ib_ja_i^{-1}b_j^{-1}$ of $\pi_1(\Sigma_g)$. Since we only have two generators, this will end up giving us the free Abelian group on four generators (it makes the relation $a_1b_1a_1^{-1}b_1^{-1}\dots a_gb_ga_g^{-1}b_g^{-1} = e$ redundant). In this case, we have the sequence of quotients $\mathbb{F}_2 \to \pi_1(\Sigma_g) \to \pi_1(\Sigma_g)/\{[a_i, b_j]\}$. All the maps are homomorphisms and as we will learn later, the Hurewicz theorem claims that the quotient of π_1 by the commutator subgroup is isomorphic to the first homology group H_1.

[2]Actually, for a connected sum $M \# N$, there exists embedded copies of $M \setminus \{p\}, N \setminus \{p\}$