Math 6510 Homework 3

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Notational Note: In any presentation of a group G, I will explicitly set the relation equal to e as opposed to simply writing a presentation like $G \cong \langle a,b,c,d | abcd^{-1}\rangle$. This is a bit untraditional, but it will remove any possible confusion.

§1.2 Problems

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Problem. *Let $X \subset \mathbb{R}^m$ be the union of convex open sets $X_1, \cdots, X_n$ such that $X_i \cap X_j \cap X_k \neq \forall i,j,k$. Then X is simply-connected*

Effectively, we mainly need to show that the intersection of two convex sets is convex, for then the van Kampen Theorem gives $\pi_1(X, x_0) = 0, \forall x_0 \in X$. First, let's establish that X is path-connected so that up to isomorphism, our computation of $\pi_1(X, x_0)$ is independent of basepoint. Each X_i is path-connected, since the definition of convex means that $\forall x, y \in X_i, tx + (1-t)y \in X_i$. Since $X_i \cap X_j \neq 0, \forall i, j$, then a path between $x \in X_i, \hat{x} \in X_j$ is simply $\gamma * \hat{\gamma}$, where $\gamma(t) = tx + (1-t)y, \hat{\gamma}(t) = ty + (1-t)\hat{x}$ for $y \in X_i \cap X_j$.

Now suppose that $X, Y \subset \mathbb{R}^m$ are convex sets. Now let $x, y \in X \cap Y$. Then by the definition of convexity, it is clear that $tx + (1-t)y \in X$ and $tx + (1-t)y \in Y$ implies $tx + (1-t)y \in X \cap Y$. This implies that $\pi_1(X_i \cap X_j \cap X_k) = 0$ and as such the van Kampen Theorem gives $\pi_1(X, x_0) = *_i \pi_1(X_i, x_0) = 0$.

3

Problem. *Show that the complement of a finite set of points in $\mathbb{R}^n$ is simply-connected if $n \geq 3$.*

From example 1.15, we know that $\mathbb{R}^n \setminus \{x\}$ is homeomorphic to $S^{n-1} \times \mathbb{R}$ (both are path-connected). In fact, since $\mathbb{R}^n \setminus \{x\} \cong \mathcal{B}(x,r) \setminus \{x\} \cong S^{n-1} \times \mathbb{R}$, where $\mathcal{B}(x,r)$ is a ball of radius r centered at x, we can use an inductive argument to show that $\mathbb{R}^n \setminus \{x_1, x_2, x_3, \ldots, x_k\} \cong \bigvee_{i=1}^k S^{n-1} \times \mathbb{R}$. Example 1.15 is the base case, so now assume that $\mathbb{R}^n \setminus \{x_1, \ldots, x_{k-1}\} \cong \bigvee_{i=1}^{k-1} S^{n-1} \times \mathbb{R}$ which is path-connected. Since cut points are topological invariants, this means that we are considering $\bigvee_{i=1}^{k-1}(S^{n-1} \times \mathbb{R}) \setminus \{(x,y)\}$, where $x \in S^{n-1}, y \in \mathbb{R}$. For $n \geq 3, S^{n-1} \setminus \{x\}$ is homeomorphic [1] to $\mathbb{R}^{n-1}$ so this implies the the set $A_y = \{(z,y) : z \in S^{n-1}\}$ is homeomorphic to a copy of $\mathbb{R}^{n-1}$ which is homotopy equivalent to a point. By contracting A_y, we have constructed the wedge sum of $S^{n-1} \times \mathbb{R}_{>y} \cup \{y\}$ and $S^{n-1} \times \mathbb{R}_{<y} \cup \{y\}$, by identifying the copies of y (Note that $\mathbb{R}_{>y} = \{x \in \mathbb{R} : x > y\}, \mathbb{R}_{<y} = \{x \in \mathbb{R} : x < y\}$). This is homeomorphic to $(S^{n-1} \times \mathbb{R}) \vee (S^{n-1} \times \mathbb{R})$ so that $\bigvee_{i=1}^{k-1} S^{n-1} \times \mathbb{R} \cong \bigvee_{i=1}^{k} S^{n-1} \times \mathbb{R}$.

Now from example 1.21 and path-connectivity, we have $\pi_1(\bigvee_{i=1}^k S^{n-1} \times \mathbb{R}) \cong *_{i=1}^k \pi_1(S^{n-1} \times \mathbb{R}) = *_{i=1}^k \pi_1(S^{n-1}) \times \pi_1(\mathbb{R})$. Since $n > 3$, Proposition 1.14 says that $\pi_1(S^{n-1}) = 0$ so that $\pi_1(\mathbb{R}^n \setminus \{x_1, \ldots, x_k\}) = 0$.

4

Problem. *Compute the fundamental group of the complement of n-lines that pass through the origin in $\mathbb{R}^3$*

Suppose that $X \subset \mathbb{R}^3$ is the union of n lines through the origin. It is claimed that a presentation for $\pi_1(\mathbb{R}^3 \setminus X, x_0)$ is,

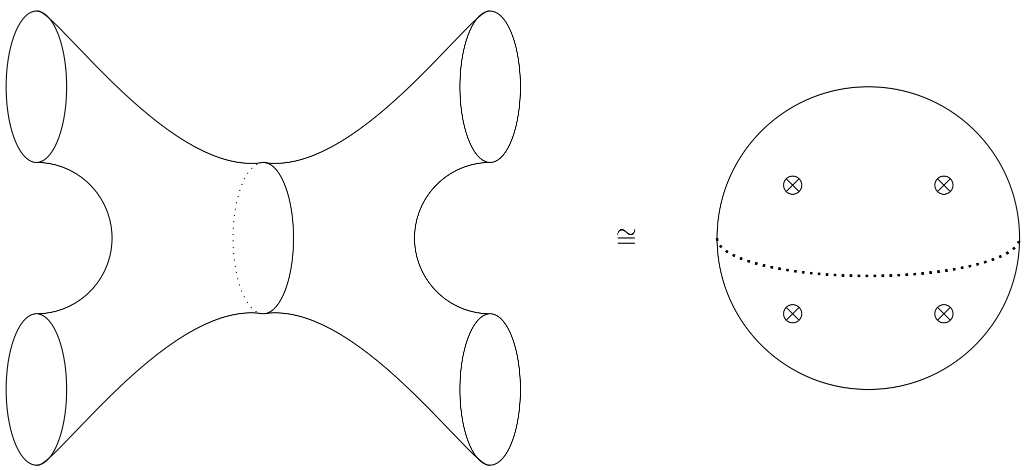
$$\pi_1(\mathbb{R}^3 \setminus X, x_0) \cong \begin{cases} \langle g_1, g_2, \ldots, g_{2n} | g_1 g_2 \cdots g_{2n} = e\rangle & \text{if } n > 1 \\ \mathbb{Z} & \text{if } n = 1 \end{cases} \tag{1}$$

Note that the choice of basepoint will not matter as the space is path-connected (This is effectively proved by the homeomorphisms in the figures on the next page.) We can prove this by induction. The base case is $\mathbb{R}^3 \setminus \{(x(t), y(t), z(t)) : t \in \mathbb{R}\}$ where $(x(t), y(t), z(t))$ is a parametrization of the line ℓ removed. Note that the intersection of the plane $\mathcal{P}_t := \{(x', y', z') : x \in \mathbb{R} \text{ and } x' = x(t), y', z' \in \mathbb{R} \text{ and } y' \neq y(t), z' \neq z(t)\}$ with $\mathbb{R}^3$ yields a punctured plane which we proved is homotopic to S^1. Note that the set of such planes is indexed by $\mathbb{R}$ and that $\{\mathcal{P}_t : t \in \mathbb{R}\} = \mathbb{R}^3 \setminus \{\ell\}$ so that

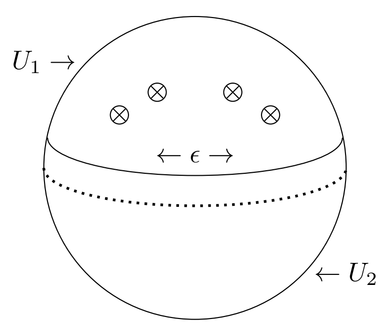
[1] Actually, this is a diffeomorphism; it's simply the stereographic projection

$\mathbb{R}^3 \setminus \ell$ which is homotopy equivalent to $S^1 \times \mathbb{R}$. Since open cylinders deformation retract to S^1 and all of the homotopy equivalent spaces are path connected, Proposition 1.18 says that $\pi_1(\mathbb{R}^3 \setminus \{\ell\}, x_0) \cong \mathbb{Z}, \forall x_0 \in \mathbb{R}^3 \setminus \{\ell\}$.

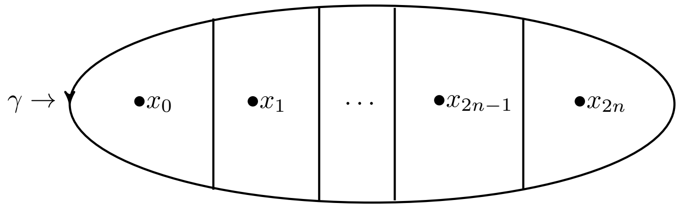
Now let's consider the higher cases. The methodology isn't exactly an inductive step (it would be a bit awkward to formulate this proof with an inductive step), but one can see via the figures provided how it would be possible to turn this into an inductive proof. Using the same logic as in the base case, we can consider a one-parameter family of planar foliations $\mathcal{P}_t$ of $\mathbb{R}^3 \setminus X$ and show that $\forall t \in \mathbb{R}$ such that $\mathcal{P}_t \neq \mathbb{R}^3 \setminus \{\vec{0}\}$, $\mathcal{P}_t$ is simply $\mathbb{R}^2 \setminus \{x_1(t), x_2(t), \ldots, x_n(t)\}$. Let $t' \in \mathbb{R}$ be such that $\mathcal{P}_{t'} = \mathbb{R}^3 \setminus \{\vec{0}\}$; such a plane must exist since the common zero locus of all the lines is the origin. This means that for $t = t', \mathcal{P}_{t'}$ is homotopy equivalent to S^1 while for $t \neq t', \mathcal{P}_t$ is homotopy to the many punctured plane. As we showed in the previous problem, this is homotopy equivalent to a bouquet of circles, $\bigvee_{i=1}^n S^1$. Finally, our punctured foliation of $\mathbb{R}^3 \setminus X$ decomposed as the disjoint union of two one-parameter families of such planes (i.e. the family of planes $\mathcal{P}_t$ with moduli $t < t', t > t'$) and a punctured plane at $t = t'$, so that we effectively have a space that is homotopy equivalent to a cylinder $S^1 \times [-\epsilon, \epsilon]$ with n-cylinders glue to each end, $S^1 \times \pm\epsilon$. Geometrically, one can envision this as the "doubled pair of pants," which is homeomorphic to a sphere with $2n$ perforations or punctures. The picture for $n = 2$ is below:



Now we can apply the van Kampen Theorem to get the presentation (1). Assume without the loss of generality that we have all of the $2n$-punctures on some single hemisphere of S^2, all of which are at least a distance (latitude) $\epsilon > 0$ away from the equator. Since changing the location of a puncture corresponds to an affine transformation of the line corresponding to the puncture (i.e. a homeomorphism!), this can be done. Let U_1 be the open hemisphere (up to the equator) that contains the $2n$-punctures and let U_2 be the union of the other open hemisphere (up to the equator) and an open annulus containing the equator up to the ϵ latitude line. In pictures, this is the following:



Now note that $U_1 \cap U_2$ is an annulus which in particular can retract to it's boundary circle. Hence $\pi_1(U_1 \cap U_2, u) \cong \mathbb{Z}, \forall u \in U_1 \cap U_2$. Since $U_2 \cong D^2$, we have $\pi_1(U_2) \cong \{0\}$. Thus we are left with the job of computing $\pi_1(U_1, \tilde{u}), \tilde{u} \in U_1$. Since $U_1 \cong \mathbb{R}^2 \setminus \{x_1, x_2, \ldots, x_{2n}\}$ we need to compute the fundamental group of the multipunctured plane. Since the translation map $\tau_{\vec{x}} : \mathbb{R}^2 \to \mathbb{R}^2, \tau_{\vec{x}}(\vec{u}) = \vec{x} + \vec{u}$ is a homeomorphism, we can assume without the loss of generality that the $2n$ punctures are on the x-axis at the points $(0, i), i \in \{1, 2, 3, \ldots, 2n\}$. Since $[0, 1]$ is contractible, we are considering the fundamental group of the following space (for the case of $n = 2$):



This clearly retracts to $\overbrace{S^1 \vee S^1 \vee \cdots \vee S^1}^{2n}$, which by Example 1.21 has the presentation $\langle g_1, \ldots, g_{2n} \rangle$, where g_i are the loops that go around the point x_i from the basepoint $\tilde{u}$. Now note that the loop denoted γ in the figure above corresponds to a loop in $U_1 \cap U_2$. This is the loop that goes around all the points $x_0, x_1, \ldots, x_{2n}$ from a basepoint $\tilde{u}$. Hence, we have: $[\gamma] = [g_1 g_2 \cdots g_{2n}]$. By the van Kampen Theorem, we mod out the subgroup generated by this equivalence class; since we know that $\pi_1(U_1 \cap U_2) \cong \mathbb{Z}$, this means that we only need to send the generator $g_1 \cdots g_{2n}$ to the trivial loop. This gives the the presentation (1).

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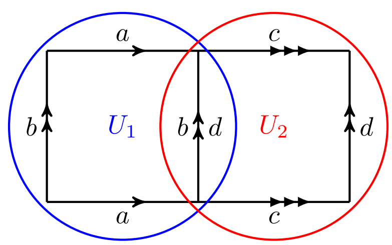
Problem. *Compute the fundamental group of two tori $S^1 \times S^1$ identified on circles $S^1 \times \{x_0\} \hookrightarrow S^1 \times S^1$*

Let X be the space of interest; since it is a quotient of two disjoint tori, it is path-connected. Formally, the 1-skeleton of this space will be the following:



As indicated in the above drawing, U_1, U_2 are neighborhoods of each 2-cells attached to this skeleton representing one torus. Under the indicated quotients, each U_1 is homotopy equivalent to a torus, so $\pi_1(U_i, x_0) = \mathbb{Z}, \forall x_0 \in \mathbb{T}^2$. Hence by van Kampen's Theorem, we have $\pi_1(X, x_0) \cong \frac{\pi_1(U_1) * \pi_1(U_2)}{N} \cong \frac{Z^2 * Z^2}{N}$ where N is the normal subgroup generated by the loop in $\pi_1(U_1 \cap U_2, x_0) \cong \mathbb{Z}$, since the loop b contracts to a circle under the quotient. This implies that bd^{-1} is trivial so that the group presentation is:

$$\pi_1(X, x_0) \cong \langle a, b, c, d | [a, b] = [c, d] = e, bd^{-1} = e \rangle \tag{2}$$

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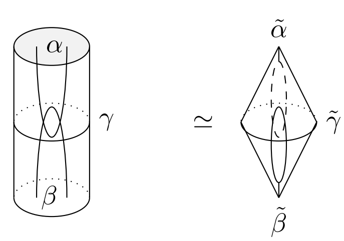
Problem. *Consider two arcs α, β embedding in $D^2 \times I$ (see figure on page 53). The loop γ is obviously null-homotopic in $D^2 \times I$ but show that there is no null-homotopy of γ in the complement of $\alpha \cup \beta$.*

Let's use a homotopy equivalence to send $D^2 \times I$ to S^2 so that this problem reduces to the case of Example 1.23. The copies of D^2 on the boundary are contractible, so that if we contract them, we get the suspension of S^1, with two interlocked loops; see the figure below.



This is homotopic to S^2 since the two cones can be resolved smoothly[2], so that we've reduced the problem to the

[2]There is a rather nice morphism of varieties in algebraic geometry called the *blow-up* which gives the explicit homotopy via a singularity resolution. Let us describe it heuristically, so that we can get an idea of how the homotopy works. The top of the cone is a singularity (the tangent space degenerates to a point), so the blow-up process "cuts" a little ϵ neighborhood around the point of the cone and glues a 2-cell by attaching the boundary to the (former) boundary of the ϵ-neighborhood. Then by relating ϵ and the radius R of the 2-cell that was glued on (i.e. something along the lines of $\epsilon \propto 2\pi R$), we can expand the ϵ-neighborhood so that as $\epsilon \uparrow h$, where h is the height of the cone, we get the upper hemisphere of S^2 and as $\epsilon \downarrow 0$ we degenerate to the cone.

case of Example 1.23. Here we see $\pi_1(X, x_0) \cong \mathbb{Z} \oplus \mathbb{Z}$ and as such γ is a non-trivial loop (geometrically, it is a meridianal loop).

17

Problem. *Show that $\pi_1(\mathbb{R}^2 \setminus \mathbb{Q}^2)$ is uncountable*

Note: I will assume knowledge of some basic analytic properties of irrational and rational numbers, such as the density of $\mathbb{Q}$ in $\mathbb{R}$.

Let $\mathbb{I}$ be the set of irrational numbers and let $X := \mathbb{R}^2 \setminus \mathbb{Q}^2$. As a set, we have

$$X = \left\{(x, y) \in \mathbb{R}^2 : x, y \in \mathbb{I}, \text{ or } x \in \mathbb{R}, y \in \mathbb{I}, x \in \mathbb{I}, y \in \mathbb{R}\right\}$$

Moreover, X is considered a topological space under the inherited subspace topology of $\mathbb{R}^2$. We have a few facts from point-set topology to establish first:

Claim. *X is path-connected*

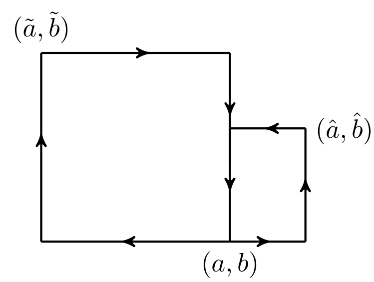
Proof. Suppose $(x, y), (x', y') \in X$ so that at least one coordinate in each element is irrational. Now suppose without the loss of generality that $x \in \mathbb{I}$. There is a path between (x, y) and an element $(a, b) \in \mathbb{I} \times \mathbb{I}$, namely the path $\gamma : I \to X, \gamma(t) = (x, y + t(b - y))$. Now if $\tilde{\gamma} : I \to \mathbb{I} \times \mathbb{I} \subset X$ is a path in $\mathbb{I} \times \mathbb{I}$ that begins at (a, b), it is claimed that $\gamma * \tilde{\gamma}$ is continuous as the two paths arrive at the same point. Since the set of irrationals is a disconnected space[3] we only require $\gamma(1) = \tilde{\gamma}(1)$ to ensure continuity. Since this was assumed in the construction of $\tilde{\gamma}, \gamma * \tilde{\gamma}$ is continuous.

Hence, we only have to show that we can construct paths in $\mathbb{I} \times \mathbb{I}$. However, this is precisely what we did before; to see this, suppose $(a, b) \in \mathbb{I} \times \mathbb{I}, (a', b') \in \mathbb{I} \times \mathbb{I}$. An explicit path between (a, b) and (a', b') is $\xi * \zeta$, where $\zeta : I \to \mathbb{I} \times \mathbb{I}$, $\xi : I \to \mathbb{I} \times \mathbb{I}$ are defined by

$$\zeta(t) = (a + t(a' - a), b) \qquad \xi(t) = (a, b + t(b' - b))$$

□

Given that X is path-connected (unlike $\mathbb{R} \setminus \mathbb{Q}$, which is totally disconnected as per the footnote), we need to be slightly more careful than one is while proving that $\pi_1(\mathbb{R} \setminus \mathbb{Q})$ is uncountable. Now let $(a, b)(\hat{a}, \hat{b}) \in X$ and let $\hat{\gamma}$ be a loop based at (a, b) that passes through $(\hat{a}, \hat{b})$. Such a loop exists, since X is path-connected. The goal is to show that for any other $(\tilde{a}, \tilde{b}) \in \pi_1(X, (a, b))$, the loop $\tilde{\gamma}$ based at (a, b) that goes through $(\tilde{a}, \tilde{b})$ is not homotopic to $\hat{\gamma}$. Based on the proof of path-connectedness, we can assume without the loss of generality[4] that $(a, b), (\hat{a}, \hat{b}) \in \mathbb{I} \times \mathbb{I}$. The picture to keep in mind is the following:



Now suppose that a homotopy of loops $H : I \times I \to X, H(s, 0) = \hat{\gamma}(s), H(s, 1) = \tilde{\gamma}(s)$ existed. Then this map would have to be continuous in both s, t. Consider the diagonal map $d : I \to I \times I, d(k) = (k, k)$. Since H is continuous, $H \circ d : I \to X$ is also continuous. However, since X is a totally disconnected space which implies that there exist closed, proper $X_0, X_1 \subset X, X = X_0 \sqcup X_1$ so that

$$I = (H \circ d)^{-1}(X) = (H \circ d)^{-1}(X_0 \sqcup X_1) = (H \circ d)^{-1}(X_0) \sqcup (H \circ d)^{-1}(X_1)$$

which implies that I is disconnected, a contradiction. Note that the above implicitly also relies on the fact that $\mathbb{R} \times \mathbb{I}, \mathbb{I} \times \mathbb{R}$ are also disconnected (although not totally disconnected).

From the first result, we know that for $x_0 \in X$, there are an uncountable number of loops based at x_0 that pass through a different irrational. From the proof above, we see that none of these loops are homotopic to each and since X is path-connected, we only need to compute $\pi_1(X, x_0)$ at one point. Hence, $\pi_1(X, x_0)$ must be uncountable.

[3] *Proof*: Let $r_1, r_2 \in \mathbb{Q}, r_1 < r_2$. The interval $(r_1, r_2) \cap \mathbb{I}$ is open in $\mathbb{I}$ since (r_1, r_2) is open in $\mathbb{R}$. The complement of $(r_1, r_2) \cap \mathbb{I}$ is $((-\infty, r_1] \cup [r_2, \infty)) \cap \mathbb{I}$. But since $r_1, r_2 \in \mathbb{Q}$, $((-\infty, r_1] \cup [r_2, \infty)) \cap \mathbb{I} = ((-\infty, r_1) \cup (r_2, \infty)) \cap \mathbb{I}$ so that (r_1, r_2) is also closed in $\mathbb{I}$. This means that there is a non-trivial clopen subset of $\mathbb{I}$, hence $\mathbb{I}$ is disconnected. In fact, $\mathbb{I}$ is *totally disconnected*, we know that given any $i \in \mathbb{I}$, there exist $r_1, r_2 \in \mathbb{Q}$ such that $\forall \epsilon > 0, \|i - r_1\|_{\mathbb{R}^2}, \|i - r_2\|_{\mathbb{R}^2} < \epsilon$ so that the set $\{i\}$ is open in $\mathbb{I}$.

[4] If $(a, b), (a', b) \in X$ and $a \in \mathbb{Q}$ and $\gamma(t) = (a + t(a' - t), b)$ then the homotopy $H(t, s) = \gamma(st)$ shows that γ is null-homotopic.

Extra Problem

Problem. *Let X be a disk, an annulus or a Möbius band, including the boundary circle or circles,*

$$\partial X \cong \begin{cases} S^1 & \text{if } X \text{ is a disk} \\ S^1 \sqcup S^1 & \text{if } X \text{ is an annulus or Möbius band} \end{cases}$$

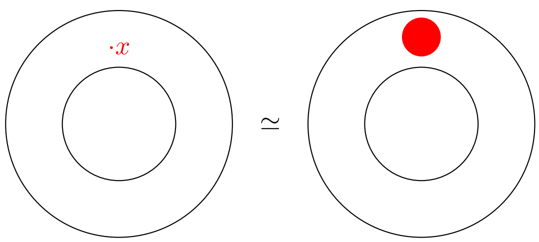
Show the following:
a) $\forall x \in X$ show that the inclusion map $X \setminus \{x\} \hookrightarrow X$ induces an isomorphism on $\pi_1(X, x_0)$ iff $x \in \partial X$
b) If Y is also a disk, annulus or Möbius Band, respectively, and if $f : X \to Y$ is a homeomorphism, show that f restricts to a homeomorphism $\partial X \to \partial Y$.
c) Show that the Möbius band is not homeomorphic to an annulus

a

($\Rightarrow$) Assume that $X \setminus \{x\} \hookrightarrow X$ induces an isomorphism $\pi_1(X \setminus \{x\}, x_0) \cong \pi_1(X, x_0)$ and suppose for a contradiction that $x \notin \partial X$. We have three cases:

1. If X is a disk this means that $X \setminus \{x\}$ is homotopy equivalent to S^1 which contradicts the assumption as the homotopy equivalent gives $0 = \pi_1(X, x_0) \cong \pi_1(X \setminus \{x\}, x_0) \cong \pi_1(S^1) = \mathbb{Z}$. This is absurd, so $x \notin \partial X$ cannot be true is X is a disk.

2. If X is an annulus, then $X \setminus x$ is not homotopy equivalent to S^1. To see this, note the following picture (red means the area was deleted):



   This shows that $X \setminus \{x\}$ has another non-trivial generator of $\pi_1(X \setminus \{x\}, x_0)$, namely the set of loops that goes around the circle that is removed. This means that $\pi_1(X \setminus \{x\}, x_0) \cong \langle a, b, c \rangle$ (i.e. each generator corresponds to a loop around either of the boundary circles or the deletion). On the other hand X deformation retracts onto either of its boundary circles (it is an *collar* neighborhood of the inner boundary circle) so that $\pi_1(X, x_0) \cong \pi_1(S^1, x_0) \cong \mathbb{Z}$. Hence x cannot be on the interior of the annulus

3. Now suppose that X is a Möbius strip. Then just as in the annulus case, $X \setminus \{x\}$ is homotopic to the removal of a circle in the interior of the Möbius Strip. This adds another non-trivial generator and as such $\pi_1(X, x_0) \ncong \pi_1(X \setminus \{x\}, x_0)$.

Hence x must be in ∂X.

($\Leftarrow$) Suppose that $x \in \partial X$. If X is a disk, then any loop in X that goes through x is homotopic to a loop that doesn't go through x (in particular, the trivial loop), so that removal of x will not affect the single (trivial) homotopy class of $\pi_1(X)$. If X is an annulus, then x is either in the outer boundary circle or the inner boundary circle. Since the annulus deformation retracts onto either of its boundary circles, we know that any loop in X that passes through x is homotopic to a loop that doesn't go through x (in particular, it is homotopic to a loop in the other boundary circle). Hence $\pi_1(X) \cong \pi_1(X \setminus \{x\})$. Finally, suppose X is a Möbius band. Since X can be defined as the unique non-trivial vector bundle over S^1 that generates $\tilde{K}(S^1)$, this simply implies that since $x \in \partial X$, X is homeomorphic to the bundle Y, where Y retracts the fibers of the base S^1 to a smaller copy of the Möbius strip. This has the effect of simply using a loop γ' in the homotopy class of a loop γ that passes through x such that γ' doesn't pass through x.

b

Suppose that we have a homeomorphism $f : X \to Y$, where X and Y are either both disks, annuli or Möbius bands. Suppose that f doesn't restrict to a homeomorphism $g : \partial X \to \partial Y$, where g would be defined as $f|_{\partial X}$. Since cut points are homeomorphism invariants, we can consider $X \setminus \{x\}, Y \setminus \{f(x)\}$Then we would have an induced isomorphism $\pi_1(X \setminus \{x\}, x_0) \cong \pi_1(Y \setminus \{f(x)\}, f(x_0))$, such that $x \in \partial X, f(x) \notin \partial Y$. But this contradicts what we showed in part (a), as we know that these fundamental groups can be isomorphic iff $x, f(x)$ are boundary points.

Now if X, Y are different, then we have to consider a few cases. If X is a disk, then Y cannot be anything other than a disk since the $\pi_1(Y)$ for Y a Möbius Strip or Annulus is different than $\pi_1(X)$. Now if X is an annulus and Y is a Möbius strip, then if such a homeomorphism existed and didn't restrict to a homeomorphism on the boundary, then the induced isomorphism $\pi_1(X, x_0) \cong \pi_1(Y, f(x_0))$. Using the fact that cut points are homeomorphism invariants, this means that we also have $\pi_1(X \setminus \{x\}, x_0) \cong \pi_1(Y \setminus \{f(x)\}, f(x_0))$ for $x \in \partial X, f(x) \notin \partial Y$, again contradicting part (a)

c

Suppose for a contradiction that there existed a homeomorphism $f : X \to Y, X$ a Möbius band, Y an annulus. This needs to restrict to a homeomorphism $g : \partial X \to \partial Y$. Since Y retracts onto it's boundary disks via retractions r_1, r_2, this implies that $f^{-1} \circ r_1, f^{-1} \circ r_2$ is a retraction of X onto it's boundary circles. We showed in last week's homework that no such retraction exists, Hatcher §1.1, problem 16(f), so that we have arrived at a contradiction.