Math 6510 Homework 5

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§0.0 Problems

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Problem. *A **deformation retraction in the weak sense** of a space X to a subspace A is a homotopy $f_t : X \to X$ such that $f_0 = \mathbb{1}_X, f_1(X) \subset A$, and $f_t(A) \subset A, \forall t$. Show that if X deformation retracts to A in this weak sense, then the inclusion $\iota : A \hookrightarrow X$ is a homotopy equivalence.*

We need to show that $\exists\, g : X \to A$ such that $g\iota \simeq \mathbb{1}_A, \iota g \simeq \mathbb{1}_X$. If $g := f_1$, where we treat f_1 as a map $A \to A$, then it is clear that $g\iota = f_1\iota : A \to A$ is homotopic to $\mathbb{1}_A$. To see this, define $\Lambda : X \times I \to X, \Lambda(x,t) = f_t\iota(x) = f_t|_A(x)$. This is map is continuous in x, t since f is a homotopy and ι is continuous. Now as this is the restriction to A of f_t (via the composition with ι), we get the desired homotopy as $\Lambda(x,0) = f_0\iota = f_0|A = \mathbb{1}_X|_A = \mathbb{1}_A, \Lambda(x,1) = f_1\iota$.

On the other hand, let's consider $\iota g = \iota f_1 : X \to X$. Let the image of f_t in X be X_t and let $\iota_1 : X_1 \hookrightarrow X$ be the natural inclusion. Now define $g_t : X_1 \to X$ by $g_t(x) = f_t \circ \iota_1$. Then $g_1 = f_1 \circ \iota_1 : X_1 \to X$ and $g_0 = \mathbb{1}_X$. If we define a homotopy $G : X_1 \times I \to X, G(x,t) = g_t(x)$, then since we have the hypotheses $f_1(X) = X_1 \subset A, f_t(X_1) \subset f_t(A) \subset A \forall t$, we then have $g_t(X_1) \subset A, \forall t$. This means that we can naturally extend G to a continuous map $G : X \times I \to X$ that serves as the desired homotopy.
$\therefore \iota$ is a homotopy equivalence

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Problem. *Show that if a space X deformation retracts to a point $x \in X$, then for each neighborhood U of $x \in X, \exists$ a neighborhood $V \subset U$ of x such that the inclusion map $V \hookrightarrow U$ is nullhomotopic*

First, let's establish a fact from Point-Set Topology, namely:

$$\text{A space } X \text{ is contractible} \iff \mathbb{1}_X \text{ is nullhomotopic.}$$

Now in our case we can see this explicitly. First, let $f_t : X \to X$ be the deformation retract in question. This means that $f_t(\{x\}) = \{x\}, \forall t, f_1(X) = \{x\}$. Note that a deformation retraction is stronger than a homotopy equivalence and from the result of the previous problem, $\exists$ a map $\iota : \{x\} \to X$ such that $f_1\iota : \{x\} \to \{x\} \simeq \mathbb{1}_{\{x\}}, \iota f_1 : X \to X \simeq \mathbb{1}_X$.

Having established this fact for our situation, let $g_t : X \to X$ be the homotopy between $g_1 = \iota f_1 : X \to X$ and $g_0 = \mathbb{1}_X$ and let U be a neighborhood of x. Since g_t is continuous in $t, \exists t_0 \in I$ such that if $t > t_0$ then $g_t^{-1}(U)$ is a strict subset of U and for $t < t_0, g_t^{-1}(U) = U$. Now let $V = g_{t_0}^{-1}(U)$ and construct a new coordinate, $s = \frac{t}{t_0}, t \in [0, t_0]$ and define the map $h : U \times I \to V, h(x,s) = g_{st_0}|_U(x)$. Note that $h(x,0) = \mathbb{1}_X, h(x,1) \subset V$. Moreover, since f_t is a deformation retraction (in the strong sense), this implies that $h_t(V) \subset V, \forall t$, so that h is a deformation retraction in the weak sense from V to $\{x\}$. From the previous problem, this implies that $\tilde{\iota} : \{x\} \to V$ is a homotopy equivalence so that $h_1\tilde{\iota}$ is homotopic to $\mathbb{1}_{\{x\}}$ (i.e. null-homotopic). To summarize the maps we have, consider the following diagram:

$$\begin{tikzcd} U \arrow[r, bend left, "g_{t_0}"] & V \arrow[l, bend left, "\iota"] \arrow[r, bend left, "h_1"] & \{x\} \arrow[l, bend left, "\tilde{\iota}"] \end{tikzcd}$$

Now since $h_1\tilde{\iota}$ is null-homotopic and $\iota : U \to V$ is homotopic to $\mathbb{1}_V$ via f, this implies that by multiplying the homotopies, $g_t * h_t$ [$*$ is the group operation of $\pi_1(X)$], we get that ι is null-homotopic.

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Problem. *a) Let X be the subspace of $\mathbb{R}^2$ consisting of horizontal segments of $[0,1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0, 1-r]$ for $r \in \mathbb{Q} \cap [0,1]$. Show that X deformation retracts to any point in the segment $[0,1] \times \{0\}$ but not to any other point.*

b) Let Y be the subspace of $\mathbb{R}^2$ that is the union of an infinite number of copies of X arranged in a zig-zag. Show that Y is contractible but does not deformation retract onto any point
c) Let Z be the zigzag subspace of Y homeomorphic to $\mathbb{R}$ indicated by the heavier line. Show there is a deformation retraction in the weak sense of Y onto Z but no true deformation retraction

a

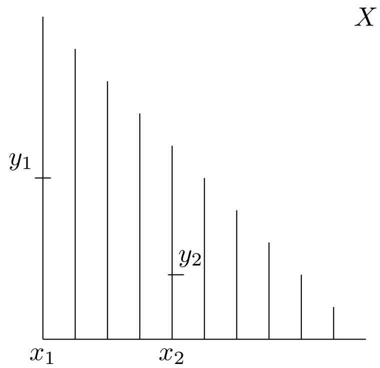
We can construct this retraction explicitly. Heuristically, it simply retracts each interval $\{r\} \times [0, 1-r]$ to $\{r\} \times \{0\}$ via a linear retraction $x \mapsto tx$. Define $h^r : X \times I \to X$ as,

$$h^r(x, y, t) = \begin{cases} (x, ty) & \text{if } (x, y) \in \{r\} \times [0, 1-r] \\ (x, y) & \text{if } (x, y) \notin \{r\} \times [0, 1-r] \end{cases}$$

This map is continuous in the first coordinate since it is stationary while the map is continuous in the second coordinate since it is simply a linear retraction of a compact interval to a point. Since $\mathbb{Q} \cap [0, 1]$ is countable and well-ordered[1], $\exists$ an order-preserving bijection $f : \mathbb{N} \cup \{0\} \to \mathbb{Q} \cap [0, 1]$ with $f(0) = 0, f(1) \uparrow \infty$. Now define a map $H : X \times I \to X$ by,

$$H(x, y, t) = \bigstar_{n \in N \cup \{0\}} h^{f(n)}(x, y, t)$$

In order for this to be a deformation retraction, we first need to show that this is well-defined as a product and that it is continuous in x, y, t.



Well-Defined:
By construction, $H^{(x,y)}(t) = H(x, y, t)$ is non-constant only if $(x, y) \in \mathbb{Q} \times [0, 1]$ and $t \in \left[2^{-n_x}, 2^{-n_x - 1}\right]$, where $n_x := f^{-1}(x, y)$. As such, there are no convergence issues since these maps are only non-constant on compact intervals. Moreover, the ordering of the bijection f ensures that the intervals patch together correctly so that $\lim_{t \uparrow 2^{-n_x}} H^{(x,y)}(t) = \lim_{t \downarrow 2^{-n_x}} H^{(x,y)}(t)$.
Continuity:
Now for fixed t, $H_t(x, y)$ is trivially continuous, since H_t is constant in the first coordinate and is a contraction in the second coordinate, just as before. As such we only need to verify that $H^{(x,y)}(t)$ is continuous. Since our construction of H is inductive, we simply need to show that for arbitrarily small $\epsilon > 0$, $x \in (1-\epsilon, 1]$, $H(x, y, t)$ is continuous in t. Since there is an induced metric[2] $\tilde{g}$ on X, we can use the conventional definition of continuity. In fact, note that this induced metric is simply a modified L^0 metric, which adds the net change in x and the net change in y [see figure]. That is,

$$\tilde{g}\left((x_1, y_1), (x_2, y_2)\right) = \delta_{x_1, x_2}|y_1 - y_2| + (1 - \delta_{x_1, x_2})\left(|x_1 - x_2| + |y_1 + y_2|\right)$$

Now let $\epsilon' \in (0, 1)$ and note that if $x_1 = x_2$ then the maximum difference in H is bounded by 1, so we only consider the case $x_1 \neq x_2, x_1, x_2 \in (1 - \epsilon, 1]$:

$$\begin{aligned} \tilde{g}\left(H(x_1, y_1, t_1), H(x_2, y_2, t_2)\right) &< \tilde{g}\left((x_1, t_1 y_1), (x_2, t_2 y_2)\right) \\ &= |x_1 - x_2| + |t_1 y_1 + t_2 y_2| < \epsilon + 2|t_1 + t_2| \\ &= \epsilon + 2|t_1 - t_2 + 2t_2| \leq 5\epsilon + 2|t_1 - t_0| \end{aligned}$$

[1] It only contains positive rationals, so that the natural ordering on $\mathbb{Q}$ induces a well-ordering
[2] If $\iota : X \to \mathbb{R}^2$ is the embedding of X into $\mathbb{R}^2$, the induced metric is $\iota^*(\delta_{ij})$

Hence if we choose $\delta < \frac{\epsilon' - 5\epsilon}{2}$, we have $|t_1 - t_2| < \delta \Rightarrow \tilde{g}\left(H(x_1, y_1, t_1) - H(x_2, y_2, t_2)\right) < \epsilon'$.
$\therefore H$ is a deformation retraction onto $[0,1] \times \{0\}$, since $H|_{[0,1]\times\{0\}\times[0,1]}$ is constant.
$\therefore X$ retracts to any point in $[0,1] \times \{0\}$ since $[0,1]$ is contractible.

Now let's argue that there are no other deformation retracts to points $x \in X$. First suppose that $f_t : X \to X$ is a deformation retract of X onto $\{x\}, x \notin [0,1] \times \{0\}$. Since X is path-connected but $X \setminus [0,1] \times \{0\}$ is not path-connected (and has countably many components), f_t must restrict to the identity on $[0,1] \times \{0\}$ unless it contracts to a point in $[0,1] \times \{0\}$ [due to the result of problem 5]. But if f_t restricts to the identity $[0,1] \times \{0\}, \forall t$, then it clearly cannot retract to any point in $X \setminus ([0,1] \times \{0\})$.

b

Now as Y is made up of an infinite copies of X, labelled $\{X_i\}_{i\in\mathbb{Z}}$. More precisely, we can define $Y = \sqcup_{i\in\mathbb{Z}} X_i / \sim$ where the equivalence relation is $(r,0) \in X_i \sim (0, 1-r) \in X_{i+1}$. Note that there is a natural inclusion $X_i \hookrightarrow Y, \forall i$. As such, we can define the central zigzag as $\{(x,0) \in X_i : i \in \mathbb{Z}, x \in [0,1]\}$. Now we can define a (weak) deformation retraction (proved in part c) onto the central zigzag, $M : Y \times I \to Y$ by $M(x,t) = \mathop{\bigstar}_{i\in\mathbb{Z}} H^i(x,t)$, where $H^i(x,t)$ is equal to $H(x,t)$ on X_i and the identity on all other X_i. The proof of continuity and well-definedness is precisely the same as a) since the countable product of homotopies constructed as a countable product of smaller homotopies is a countable product. As such, once we retract to the central zig-zag with $M(x,1)$, we can contract the zig-zag to a point proving that X is contractible. This can be formalized as in the previous part by an inductive process, by induction on i. With a finite number of X_i, we can have a well-defined "origin" and then define M via a limiting process that contracts the X_i from left to right.

However, this is not a deformation retraction. This is because an arbitrary $x \in Y$ will be on a "leaf" on some X_i so by part a), there is no deformation retraction. Note that even if $x \in Y$ is on $[0,1] \times \{0\} \subset X_i$, for some i, then x is still on a leaf of X_{i+1}, by the construction of Y so that part a) still applies.

c

It is relatively clear that the construction of M is a weak deformation retract, since M is constant on $[0,1] \times \{0\} \subset X_i, \forall i$. However, it is also clear that M is not a deformation retract for the same reason there is no deformation retraction to any point $x \in Y$. That is, all of the rational points $(r,0) \in X_i \subset Y$ are also on the leaves of X_{i+1}, so if there existed a deformation retraction, we could restrict it to a leave to get a deformation retraction to a point of Y, a contradiction (due to b).