Math 6510 Homework 6

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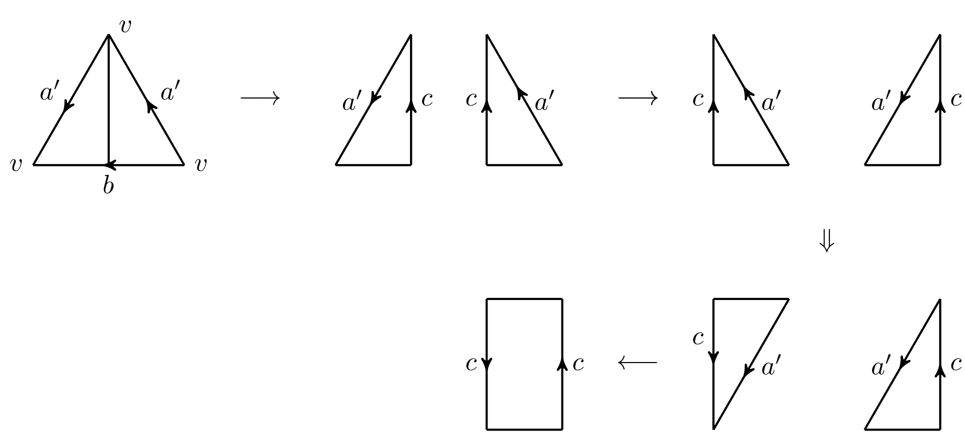
§2.1 Problems

1

Problem. *What familiar space is the quotient Δ-complex of a 2-simplex $[v_0, v_1, v_2]$ obtained by identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$, preserving the ordering of the vertices?*

There are two ways of seeing that this space X is a Möbius Strip. The first way requires the fact that the 2-simplex $[v_0, v_1, v_2]$ is a compact 2-manifold with corners. It is moreover homeomorphic (but not diffeomorphic) to a compact 2-manifold without corners, so we can can assume without the loss of generality that X is a compact 2-manifold. Since $[v_0, v_1], [v_1, v_2] \subset [v_0, v_1, v_2]$ are 1-simplices with opposite orientation, X is going to be non-orientable. Since the only non-orientable compact 2-manifolds are the Möbius Strip and the Klein Bottle, X must be one of these two manifolds. Moreover, since we are only quotienting 1 pair of 1-simplices, this means that the non-orientable compact 2-manifold must be a Möbius strip, and not a Klein bottle.

On the other hand, we can proceed purely topologically. By preserving the orientation of the 1-simplices $[v_0, v_1], [v_1, v_2]$, we must glue $[v_0, v_1]$ to $[v_1, v_2]$. Using the fundamental polygon, we can now show that X is a Möbius Strip:[1]

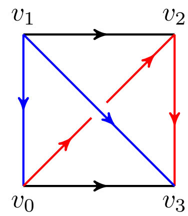


2

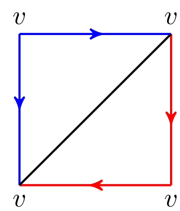
Problem. *Show that the Δ-complex obtained from Δ^3 by performing the edge identifications $[v_0, v_1] \sim [v_1, v_3]$ and $[v_0, v_2] \sim [v_2, v_3]$ deformation retracts onto a Klein Bottle*

[1] Since there is a common abuse of notation using an arrow for "gluing" and for orientation, the arrow for b is dropped so there is no confusion of its meaning of "orientation"

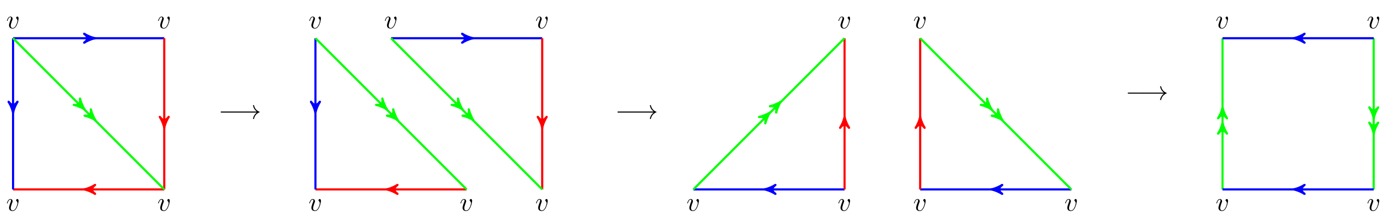
Using a projection, the quotient X of Δ^3 can be represented by the 1-skeleton (with identifications colored),



Now from problem 1, we see that the faces $[v_0, v_3, v_1], [v_0, v_3, v_2]$ are homeomorphic to Möbius Strips that are joined together along the 1-simplex $[v_0, v_3]$. Since Δ^3 deformation retracts onto it's faces, it will retract onto $[v_0, v_3, v_1] \cup [v_0, v_3, v_2]$. This leaves us with a fundamental polygon,



By rearranging this polygon, much as we did with the polygon in problem 1, we will get the fundamental polygon of a Klein Bottle. Explicitly, this is done as follows:



3

Problem. *Construct a Δ-complex structure on $\mathbb{RP}^n$ as a quotient of a Δ-complex structure on S^n having vertices the two vectors of length 1 along each coordinate axis in $\mathbb{R}^{n+1}$.*

The intuitive idea behind the Δ-complex structure on S^n is that we want to put an n-simplex in each "quadrant" of $\mathbb{R}^{n+1}$ and then glue the boundaries accordingly. If $\{\hat{e}_i\}_{i=1}^{n+1}$ are the basis vectors of $\mathbb{R}^{n+1}$, let $v_i^{\pm} = \pm\hat{e}_i$ be the vertices of this Δ-complex. Now note that for $\mathbb{R}^{n+1}$ we have 2^n "quadrants" as there are 2^{n+1} combinations of the orientations[2] $|\hat{e}_0^{\pm}\hat{e}_1^{\pm}\cdots\hat{e}_{n+1}^{\pm}|$. Now for each of these combinations, attach an n-simplex. For example if $n = 2$, then we attach 8 2-simplices, $[v_0^+, v_1^+, v_2^+], [v_0^+, v_1^+, v_2^-], [v_0^+, v_1^-, v_2^-], [v_0^+, v_1^-, v_2^+], [v_0^-, v_1^+, v_2^+], [v_0^-, v_1^-, v_2^+], [v_0^-, v_1^+, v_2^-], [v_0^-, v_1^-, v_2^-]$ to their respective basis vectors in $\mathbb{R}^3$. It is clear that this construction of 2^{n+1} n-simplices will give a space homeomorphic to S^n. Now to get a Δ-structure on $\mathbb{RP}^n$, we simply need to identify opposite simplices. That is, $v \sim -v$ for n-simplices v. For example, if $n = 2$, then under this quotient we have,

$$[v_0^+, v_1^+, v_2^+] \sim [v_0^-, v_1^-, v_2^-], [v_0^+, v_1^+, v_2^-] \sim [v_0^-, v_1^-, v_2^+], [v_0^+, v_1^-, v_2^+] \sim [v_0^-, v_1^+, v_2^-], [v_0^-, v_1^+, v_2^+] \sim [v_0^+, v_1^-, v_2^-]$$

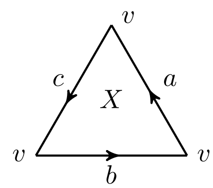
Note that $-v$ simply reverses *all* the basis vectors.

4

Problem. *Compute the simplicial homology groups of the triangular parachute obtained from Δ^2 by identifying it's three vertices to a single point*

[2] I am using $|\cdot|$ to denote orientation of a set of basis vectors, since $[\cdot]$ is already used for simplices

We have the fundamental polygon,



We have 1 face, 3 edges and 1 vertex so that $\Delta^2(X), \Delta^0(X) \cong \mathbb{Z}, \Delta^1(X) \cong \mathbb{Z}^3$. Note that $\partial_2(X) = b + a - c, \partial_1(a) = \partial_1(b) = \partial_1(c) = \partial_0(v) = 0$. Hence $\ker \partial_2 = 0, \ker \partial_1 = 0, \ker \partial_0 = 0$. On the other hand, $\operatorname{Im} \partial_2 = \mathbb{Z}$ as the subgroup $\langle b + a - c\rangle \lhd \langle a, b, c\rangle$ is free on on generator[3]. Hence we have,

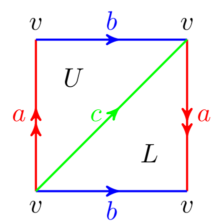
$$H_2^\Delta(X) = \ker \partial_2 \cong \{0\}, H_1^\Delta(X) = \ker \partial_1 / \operatorname{Im} \partial_1 \cong \mathbb{Z}^3 / \mathbb{Z} \cong \mathbb{Z}^2, H_0^\Delta(X) = \ker \partial_0 = \mathbb{Z}$$

Since this is clearly a 2-manifold with corners, $H_n^\Delta(X) = \{0\}, \forall n \geq 3$.

5

Problem. *Compute the simplicial homology groups of the Klein bottle using the Δ-complex structure described at the beginning of this section*

Recall from page 102 of Hatcher that the Klein Bottle K has the following Δ-complex structure:



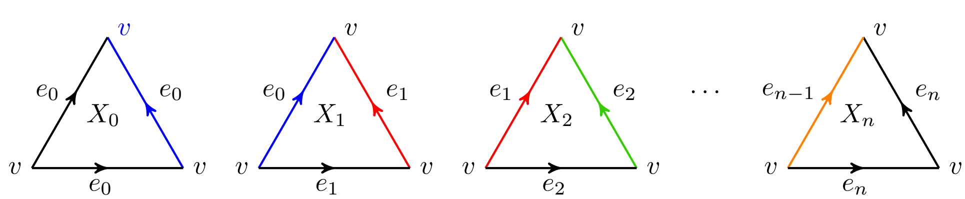
We have 2 faces, 3 edges and 1 vertex so that $\Delta^2(K) \cong \mathbb{Z}^2, \Delta^1(K) \cong \mathbb{Z}^3, \Delta^0(K) \cong \mathbb{Z}$. Now note that $\partial_2(pU + qL) = p\partial_2(U) + q\partial_2(L) = p(a + b - c) + q(a - b + c) = (p + q)a + (p - q)(b - c)$ so that $\ker \partial_2 = \{0\}$ since $p + q, p - q = e \Rightarrow p = q = e$. Note that $\operatorname{Im} \partial_2 = \{(p + q)a + (p - q)(b - c) | p, q \in \mathbb{Z}\}$. Moreover, $\partial_1(a) = \partial_1(b) = \partial_1(c) = \partial_0(v) = e$ so that $\ker \partial_1 = \mathbb{Z}^3, \ker \partial_0 = \mathbb{Z}, \operatorname{Im} \partial_1 = \operatorname{Im} \partial_0 = \{0\}$. Now note that $H_1^\Delta(K) = \ker \partial_2 / \operatorname{Im} \partial_2$ so that in the quotient $(p + q)a + (p - q)(b - c) = e$. In particular, if $p = q = 1$, we have $2a = e$, so that there is non-trivial torsion. Moreover, if $p = 1, q = -1$, we find that $b - c = e$. Hence we have the presentation, $H_1^\Delta(K) \cong \langle a, b, c | 2a = b - c[a, b] = [b, c] = [a, c] = e\rangle \cong \mathbb{Z} \times \mathbb{Z}_2$. In sum, the homology groups are:

$$H_n^\Delta(K) = 0, \forall n \geq 2, H_1^\Delta(K) = \mathbb{Z} \times \mathbb{Z}_2, H_0^\Delta(K) = \mathbb{Z}$$

6

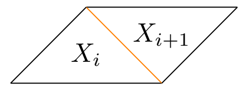
Problem. *Compute the simplicial homology groups of the Δ-complex obtained from $n + 1$ 2-simplices $\Delta_0^2, \cdots, \Delta_n^2$ by identifying all three edges of Δ_0^2 to a single edge, and for $i > 0$ identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$ of Δ_i^2 to a single edge and the edge $[v_0, v_2]$ to the edge $[v_0, v_1]$ of Δ_{i-1}^2.*

Let's start by trying to get some intuition for the space we are working with. Consider the $n + 1$ 2-simplices, as drawn below:



[3]I am using the fact that subgroups of free groups are free

Now if we glue any two adjacent 2-simplices, X_i, X_{i+1}, then we get,



Now let's compute the homology groups. We have 1 vertex, $n+1$ edges and $n+1$ faces so that $\Delta^0(X) \cong \mathbb{Z}, \Delta^1(X), \Delta^2(X) \cong \mathbb{Z}^{n+1}$. Note that $\partial_0(v) = \partial_1(e_i) = e$ so that $\ker \partial_0 = \mathbb{Z}, \operatorname{Im} \partial_1 = e, \ker \partial_1 = \mathbb{Z}^{n+1}$ and as such $H_0^\Delta(X) \cong \mathbb{Z}$. Now let's compute $\ker \partial_2$; note that:

$$\partial_2 X_i = \begin{cases} e_0 & \text{if } i = 0 \\ 2e_i - e_{i-1} & \text{if } i = 1 \end{cases} \tag{1}$$

Hence for an arbitrary 2-subsimplex $\sum_i a_i X_i$, we have $\partial_2 \left(\sum_i a_i X_i\right) = e \iff a_n = 0$, since the only term with e_n is $\partial_2 X_n$. But then if $a_n = 0, \partial_2 \left(\sum_{i=1}^n a_i X_i\right) = \partial_2 \left(\sum_{i=1}^{n-1} a_i X_i\right) = e \iff a_{n-1} = 0$ since the only term containing e_{n-1} will be X_{n-1}. Continuing this argument inductively, we see that $\sum_i a_i X_i \in \ker \partial_2 \iff a_i = 0, \forall i$. Hence $H_2^\Delta(X) = \ker \partial_2 = e$. Hence the only thing left to compute is $\operatorname{Im} \partial_2$. From (1), it is clear that a basis for $\operatorname{Im} \partial_2 = \{e_0\} \cup \{2e_i - e_{i-1} : 1 \leq i \leq n\}$. Finally, note that in $H_1^\Delta(X) = \ker \partial_1 / \operatorname{Im} \partial_2$, we set $e_0 = e$ and $2e_i - e_{i-1} = e$ so that $e_0 = 0, 2e_i = e_{i-1}$. This implies that $2e_1 = e_0 = 0, 2^2 e_2 = e_0 = 0, \ldots 2^k e_k = e_0 = 0$ so that *all* of the edges represent torsion elements. Summarizing the results, we have:

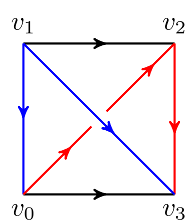
$$\begin{aligned} &H_0^\Delta(X) \cong \mathbb{Z} \\ &H_1^\Delta(X) \cong \mathbb{Z}^{n+1} / (\mathbb{Z} \times 2\,\mathbb{Z} \times \cdots 2^n\,\mathbb{Z}) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^2} \times \cdots \times \mathbb{Z}_{2^n} \\ &H_n^\Delta(X) \cong \{e\}, n \geq 2 \end{aligned}$$

Note that the second isomorphism in the expression for $H_1^\Delta(X)$ comes from taking projections on each coordinate.

7

Problem. *Find a way of identifying pairs of faces of Δ^3 to produce a Δ-complex structure on S^3 having a single 3-simplex, and compute the simplicial homology groups of this Δ-complex*

Recall that a projection of the 1-skeleton of Δ^3 is,



Now consider the relation $\sim$ defined by $\sigma_1 := [v_0, v_1, v_3] \sim [v_0, v_2, v_3], \sigma_2 := [v_1, v_2, v_3] \sim [v_0, v_1, v_2]$. This equivalence relation induces relations on 1-simplices:

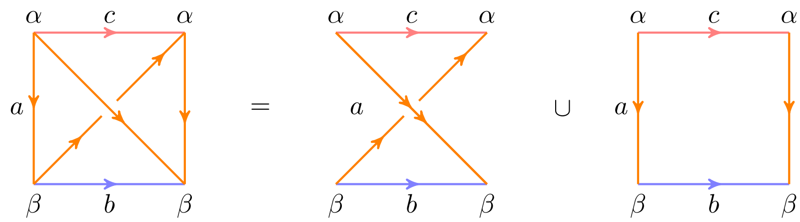
$$\begin{aligned} \text{First Relation: } &[v_2, v_3] \sim [v_1, v_3], [v_0, v_1] \sim [v_0, v_2] \\ \text{Second Relation: } &[v_0, v_1] \sim [v_1, v_3], [v_0, v_2] \sim [v_2, v_3] \end{aligned}$$

Combining the equivalence classes under the two relations gives three classes:

$$a = \{[v_0, v_1], [v_0, v_2], [v_1, v_3], [v_2, v_3]\}, b = \{[v_0, v_3]\}, c = \{[v_1, v_2]\}$$

On the other hand, this relation give $\alpha = \{[v_0] \sim [v_3]\}, \beta = \{[v_1] \sim [v_2]\}$. Hence we have 2 vertices, 3 edges, 2 faces and 1 3-simplex so $\Delta_0(X) \cong \mathbb{Z}^2 \cong \langle \alpha, \beta | [\alpha, \beta] = e \rangle$, $\Delta_1(X) \cong \mathbb{Z}^3 \cong \langle a, b, c | [a_i, a_j] = e, \forall i, j \rangle$, $\Delta_2(X) \cong \mathbb{Z}^2 \cong \langle \sigma_1, \sigma_2 : [\sigma_1, \sigma_2] \rangle$ and $\Delta_3(X) \cong \mathbb{Z} \cong \langle A \rangle$. Now let's validate the claim that $X = \Delta^3 / \sim$ is homeomorphic to S^3.

It is *not* immediately obvious that $X \cong \Delta^3/\partial\Delta^3$, since we still have three distinguished edges under the quotient. As such, we can proceed by considering the definition of $S^3 \cong S^1 \times D^2 \cup D^2 \times S^1$, where the two solid tori have boundary circles that are glued along their boundaries. Under the identification, the projection of the 1-skeleton becomes,



The second polygon on the right looks very much the fundamental polygon of $\mathbb{T}^2$. However, note that the quotient is orientation-preserving so the lines of a must all have the same orientation under the quotient. This means that the first polygon on the right (which when untwisted, initially looks like the fundamental polygon of $\mathbb{RP}^2$) is forced to have the orientation of the second polygon. As such, it *also* represents a torus under the quotient. The "twistedness" in the above diagram stems from the fact that the two tori in S^3 are mirror images of each other, so that relative to the global frame[4] of one torus (i.e. the second polygon on the right), the other torus looks as if it has the opposite orientation. Hence the above figure represents the fundamental polygon of two solid tori glued along their boundaries.

Now let's compute the homology of this space. Let $A = [\tilde{v}_0, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3]$ be the generator of $\Delta_3(X)$ (the tilde denotes image under quotient). Then note that $\partial_3(X) = 0$ and $\ker \partial_3(X) = \mathbb{Z}$, since

$$\begin{aligned}
\partial_3(A) &= A|_{[\tilde{v}_1,\tilde{v}_2,\tilde{v}_3]} - A|_{[\tilde{v}_0,\tilde{v}_2,\tilde{v}_3]} + A|_{[\tilde{v}_0,\tilde{v}_1,v_3]} - A|_{[\tilde{v}_0,\tilde{v}_1,\tilde{v}_2]} \\
&= A|_{[v_1,v_1,v_0]} - A|_{[v_0,v_1,v_0]} + A|_{[v_0,v_1,v_0]} - A|_{[v_0,v_1,v_1]} \\
&= \sigma_1 - \sigma_2 + \sigma_2 - \sigma_1 \\
&= 0
\end{aligned}$$

Hence $H_3^\Delta(X) = \mathbb{Z}$. Now let's look at ∂_2:

$$\begin{aligned}
\partial_2(n_1\sigma_1 + n_2\sigma_2) &= n_1\left(-a - b + a\right) + n_2\left(c - a + a\right) \\
&= -n_1\sigma_1|_b + n_2(\sigma_2|_c) \\
&= n_2 c - n_1 b
\end{aligned} \tag{2}$$

Hence $\ker \partial_2 = 0$ so $H_2^\Delta(X) = 0$. It is clear that $\partial_0 = 0$ and $\ker \partial_0 = \mathbb{Z}^2$, so we only need $\operatorname{Im}\partial_1, \operatorname{Im}\partial_2, \ker\partial_1$. Note that $\partial_1(a) = \beta - \alpha, \partial_1(b) = 0, \partial_1(c) = 0$. Since a basis for $\Delta_0(X)$ is $\{\alpha, \beta - \alpha\}$, $\operatorname{Im}\partial_1 \cong \mathbb{Z}, \ker\partial_1 \cong \mathbb{Z}^2$ and $H_0^\Delta(X) \cong \mathbb{Z}$. However from (2), $\operatorname{Im}\partial_2$ is generated by $\{b, c\}$ so $H_1^\Delta(X) \cong \ker\partial_1/\operatorname{Im}\partial_1 \cong \langle b, c | [b,c]\rangle / \langle b, c | [b,c]\rangle = 0$. In summary,

$$H_i^\Delta(X) \cong \begin{cases} \mathbb{Z} & \text{if } i \in \{0,3\} \\ 0 & \text{else} \end{cases} \tag{3}$$

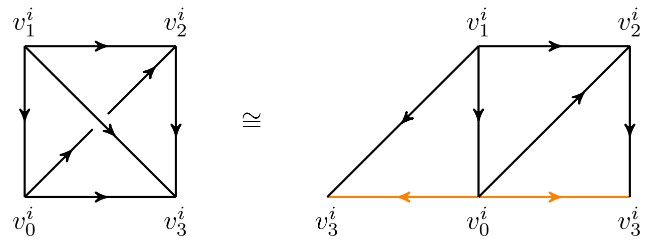
as expected.

8

Problem. *Construct a 3-dimensional Δ-complex X from n tetrahedra $T_1, \dots, T_n$ by the following two steps. First arrange the tetrahedra in a cyclic pattern as in the figure on page 131, so that each T_i shares a common vertical face with it's two neighbors T_{i-1}, T_{i+1}, subscripts being taken mod n. Then identify the bottom face of T_i with the top face of T_{i+1} for each i. Show the simplicial homology groups of X in dimensions $0, 1, 2, 3$ are $\mathbb{Z}, \mathbb{Z}_n, 0, \mathbb{Z}$, respectively. This is an example of a Lens Space.*

[4] I mean "frame" in the mathematical sense of Cartan, not a physical sense. Recall that the $\mathbb{T}^2$ has a globally defined orientation form inherited from each copy of S^1 and moreover that $\mathbb{T}^2$ is parallelizable and Ricci-flat, so that it admits a global frame $(\hat{e}^1, \hat{e}^2)$. The mirror image concept comes from strict orientation preservation in the quotient of Δ^3. If orientation wasn't preserved, this could not be S^3, since S^3 is orientable.

Let's start with the tetrahedron $T^i = [v_0^i, v_1^i, v_2^i, v_3^i]$. Using colors to denote equivalence relations, we can draw the 1-skeleton in a more useful way for this problem:



Let's first start with gluing the "left and right faces" of the tetrahedron to it's neighbors. We define the left face to be the 2-simplex $[v_0^i, v_1^i, v_3^i]$ and the right face to be $[v_0^i, v_2^i, v_3^i]$. Then our quotient simply sets $[v_0^i, v_1^i, v_3^i] \sim [v_0^{i+1}, v_2^{i+1}, v_3^{i+1}], [v_0^i, v_2^i, v_3^i] \sim [v_0^{i-1}, v_1^{i-1}, v_3^{i-1}]$. Using the inclusion-exclusion principle, we can compute the number of vertices, edges and faces at this point.

Vertices
Before quotienting have $4n$ vertices. From the definition of the quotient, we see that $[v_0^i, v_3^i] \sim [v_0^j, v_3^j], \forall i, j$ so that the $2n$ vertices $\{v_0^i, v_3^i : 1 \leq i \leq n\}$ become 2 vertices giving $2n + 2$ vertices. Now we have $v_1^i \sim v_2^{i+1}$, so that the $2n$ vertices $\{v_1^i, v_2^{i+1} : 1 \leq i \leq n\}$ become n vertices. Hence we have $n + 2$ vertices.

Edges
Before quotienting we have $6n$ edges.Using $[v_0^i, v_3^i] \sim [v_0^j, v_3^j], \forall i, j$, we see that n edges become 1 edge so that we have $5n + 1$ edges. Since $[v_0^i, v_1^i] \sim [v_0^{i+1}, v_2^{i+1}]$, we have $2n$ edges becoming n edges. The same holds for the relation $[v_2^i, v_3^i] \sim [v_1^{i-1}, v_3^{i-1}]$ so that we have $3n + 1$ edges.

Faces
Before quotienting we have $4n$ faces. Geometrically, one can see that we only lose n faces since we are gluing the left and right faces in a cycle. Hence we have $3n$ faces.

Now we need to do glue the bottom face of T^i to the top face of T^{i+1}. We will define the top face to be $[v_1^i, v_2^i, v_3^i]$ and the bottom face to be $[v_0^i, v_1^i, v_2^i]$ so that our relation becomes $[v_0^i, v_1^i, v_2^i] \sim [v_1^{i+1}, v_2^{i+1}, v_3^{i+1}]$. Let's again compute the effect on vertices, edges and faces.

Vertices
This quotient sets $v_3^{i+1} \sim v_0^i$. Since $[v_0^i, v_3^i] \sim [v_0^{i+1}, v_3^{i+1}]$, this implies that we lose one vertex, giving us $n + 1$ vertices. Moreover, we now glue v_1^i to v_1^{i+1}, v_2^i to v_2^{i+1}. From before, we know that $v_1^i \sim v_2^{i+1}$ so that we have $v_1^{i+1} \sim v_1^i \sim v_2^{i+1} \sim v_2^i, \forall i$. Hence we send the n vertices $\{v_2^i : 1 \leq i \leq n\} \sim \{v_1^i : 1 \leq i \leq n\}$ to 1 vertex so that we are left with 2 vertices. Denote these vertices $\alpha := v_0^1, \beta := v_1^1$

Edges
Since the edge $[v_0^i, v_3^i]$ has degenerated to a vertex, we lose one edge, giving $3n$ edges. Moreover, $\{[v_1^i, v_2^i] : 1 \leq i \leq n\}$ has also degenerated to a point so that we only have 1 edge

Faces
All of the faces have degenerated.

As such, we have 1 3-simplex, 0 2-simplices, 1 1-simplex and 2 0-simplices. Since this space is path-connected (as the quotient of a path-connected space), Proposition 2.7 prescribes that $H_0^\Delta(X) \cong \mathbb{Z}$. Now note that $\partial_1[\alpha, \beta] = \beta - \alpha = v_1^1 - v_0^1$.

9

Problem. *Compute the homology groups of the Δ-complex X obtained from Δ^n by identifying all faces of the same dimension. Thus X has a single k-simplex for each $k \leq n$*

Claim: Let X^n correspond to the Δ-complex obtained from Δ^n. The homology groups are

$$H_i^\Delta(X) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \text{ or } i = n \text{ if n odd} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

We will proceed by induction.

Base Case:
If $n = 0$, then there is only one vertex and it is clear that $H_0^\Delta(X^0) \cong \mathbb{Z}, H_i^\Delta(X^0) \cong \{0\}, i \neq 0$

Induction Step: Consider the space X^n. Assume that (4) holds for $k < n$. Now since X^n has only 1 k-simplex, for all $k \leq n$, $\Delta_k(X^n) \cong \mathbb{Z}, \forall k \leq n$. Now let a_k be the generator of $\Delta_k(X^n)$. From the definition of the boundary operator, we have

$$\begin{aligned}
\partial a_k &= \sum_{i=1}^{n+1} (-1)^i a_{k-1} \\
&= \begin{cases} a_{k-1} & \text{if n+1 odd} \\ 0 & \text{if n+1 even} \end{cases} \\
&= \begin{cases} a_{k-1} & \text{if n even} \\ 0 & \text{if n odd} \end{cases}
\end{aligned}$$

Hence if k is even, $\ker \partial a_k = \{e\}$ and if k is odd, then $\ker \partial a_k \cong \mathbb{Z}$. Now note that since $H_k^\Delta(X^n) \cong \ker \partial_k / \operatorname{Im} \partial_{k+1}$, if k is even, then $\ker \partial_k$ is trivial and $H_k^\Delta(X^n) \cong \{e\}$. On the other hand, if k is odd and not equal to n, then $k+1$ is even so $\operatorname{Im} \partial_{k+1} \cong \mathbb{Z}$ and $H_k^\Delta(X^n) \cong \ker \partial_k / \operatorname{Im} \partial_{k+1} \cong \mathbb{Z}/\mathbb{Z} \cong \{e\}$. Finally if $k = n$ and n odd, then $\partial_{n+1} = 0$, so $H_n^\Delta(X^n) \cong \ker \partial_n \cong \mathbb{Z}$.

Additional Problems

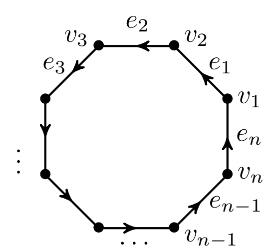
A1

Problem. *Compute the simplicial homology groups of S^1 with the Δ-complex structure having n vertices and n edges, all the edges being oriented in the same direction around the circle*

We need to show that,

$$H_i^\Delta(S^1) = \begin{cases} \mathbb{Z} & \text{if } i \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases}$$

The Δ-complex in question can be represented by something that looks similar to the fundamental polygon of the genus-g surface:



By construction we have $\Delta_0(S^1) \cong \mathbb{Z}^n, \Delta_1(S^1) \cong \mathbb{Z}^n$. Note that $\partial_0 v_i = 0, \forall i$, so $\ker \partial_0 \cong \mathbb{Z}^n$. On the other hand,

$$\partial_1 e_i = \begin{cases} v_{i+1} - v_i & \text{if } i \neq n \\ v_1 - v_n & \text{if } i = n \end{cases}$$

Hence for an arbitrary element ξ of $\Delta_1(S^1), \xi = \sum_{i=1}^n a^i e_i$, we have,

$$\begin{aligned}
\partial_1 \xi = \partial_1 \left(\sum_{i=1}^n a^i e_i \right) &= \sum_{i=1}^n a^i \partial e_i \\
&= \sum_{i=1}^{n-1} a^i (v_{i+1} - v_i) + a^n (v_1 - v_n) \\
&= (a^n - a^1) v_1 + (a^1 - a^2) v_2 + \cdots + (a^{n-1} - a^n) v_n
\end{aligned}$$

Now if $\xi \in \ker \partial_1$, then $a^i - a^{i+1} = e$ if $1 \leq i \leq n-1$ and $a^n - a^1 = e$. This means that $a^1 = a^2 = a^3 = \cdots = a^n$. If any of the a^i is zero, then this forces $a^j = 0, \forall j$ so that any non-trivial element in $\ker \partial_1$ must contain all of the edges. Hence any element of $\ker \partial_1$ is of the form $k \sum_{i=1}^n e_i, k \in \mathbb{Z}$. This implies that $\sum_{i=1}^n e_i$ generates $\ker \partial_1$ and as such $H_1^\Delta \cong \ker \partial_1 \cong \mathbb{Z}$.

On the other hand, it is claimed that $\langle v_1 - v_2, v_2 - v_3, v_3 - v_4, \ldots, v_{n-1} - v_n \rangle$ forms a basis for $\operatorname{Im} \partial_1$. This follows directly from the fact that $\sum_i v_i - v_{i+1} = v_1 - v_n$ so the last generator from the above computation is redudant. Hence $H_0^\Delta(X) \cong \ker \partial_0 / \operatorname{Im} \partial_1 \cong \mathbb{Z}^n / \langle v_i - v_{i+1} \rangle \cong \mathbb{Z}$, since the quotient sets $v_i = v_j, \forall i,j \in \{1, \ldots, n\}$.

A2

Problem. *Regarding Δ^n as a Δ-complex in the natural way, show that if a subcomplex $X \subset \Delta^n$ has $H_{n-1}^\Delta(X)$ nonzero, then $X = \partial \Delta^n$.*

Suppose for a contradiction at such a subcomplex X with $H_{n-1}^\Delta(X) \neq \{e\}, X \neq \partial\Delta^n$ existed and let $Y = \partial \Delta^n$. Furthermore, let ∂^X, ∂^Y be the associated boundary operators. Since there are $n+1$ elements $\{a_i\}_{i=0}^n$ of $\Delta_n(\Delta^n)$, for some strict subset $J \subset \{0, \ldots, n\}$ we must have $\{a_i\}_{i \in J} \subset X$. Since each a_i is equal to $[v_0, \ldots, \hat{v}_k, \ldots, v_n]$ for some k; as such, we can assume without the loss of generality that $a_i = [v_0, \ldots, \hat{v}_i, \ldots, v_n]$. Let $k \in \{0, \ldots, n\} \setminus J$ so that $[v_0, \ldots, \hat{v}_k, \ldots, v_n] \notin X$. Such an element exists since$J$ is a strict subset of $\{0, \ldots, n\}$. Now note that

$$\partial^X[v_0, \ldots, \hat{v}_k \ldots, v_n] = \sum_{i<k} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_k, \ldots, v_n] + \sum_{i>k} (-1)^{i-1} [v_0, \ldots, \hat{v}_k, \ldots, \hat{v}_i, \ldots, v_n]$$

As such,[5] $\partial a_j \cap \partial^X[v_0, \ldots, \hat{v}_k, \ldots, v_n] \neq \emptyset, \forall j \in J$. Now note that $\ker \partial^Y \neq \{e\}$, since $\partial^Y \left(\sum_{i=0}^n (-1)^i a_i \right) = 0$. This sum vanishes because the gluing of two $n-1$ simplices (in Δ^n) along an $n-2$ simplex forces pairwise opposite orientations[6] on the two glued $n-2$ simplices. Since a " As such since $[v_0, \ldots, \hat{v}_k, \ldots, v_n] \in Y \setminus X$, we have, for each $a_j, j \in J$, $[v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_k, \ldots v_n] \in \partial^X a_j \neq 0$. This means that for any $\xi = \sum_{i \in J} n^i a_i, n^i \in \mathbb{Z}$, we necessarily have $\sum_{i \in J} n^i [v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_k, \ldots, v_n \in \partial \xi$.

$\therefore \partial^X \xi \neq \{e\}, \forall \xi \in \Delta_{n-1}(X)$
$\therefore \ker \partial^X = \{e\}$ and $H_{n-1}^\Delta(X) = \{e\}$, a contradiction

[5] By intersection, I am abusing of notation by treating ∂a_j as a set of oriented $n-1$ simplices as opposed to formal sum. Effectively, a non-empty intersection of two boundaries $\partial a_j, \partial a_k$ implies that $\partial a_k - \partial a_j = \partial(a_k - a_j) \neq \emptyset$

[6] This is the "geometric" rationale for the alternating sum in the definiton of the boundary operator