Math 6510 Homework 9

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§2.2 Problems

1

Problem. *Prove the Brouwer fixed point theorem for maps $f : D^n \to D^n$ by applying degree theory to the map $S^n \to S^n$ that sends both the northern and southern hemispheres of S^n to the southern hemisphere via f.*

We will proceed by contradiction. Intuitively, the map the sends both the northern and southern hemispheres to the southern hemisphere somehow deforms the northern hemisphere in D^{n-1} while keeping the southern hemisphere and equator fixed. Suppose $S^n \hookrightarrow \mathbb{R}^{n+1}$ via the standard definition as the zero locus of the polynomial $(\sum_i x_i^2) - 1$. Then the northern and southern hemispheres are:

$$N = \{(x_1, \ldots, x_{n+1}) : \sum_i x_i^2 - 1 = 0, x_{n+1} \geq 0\}$$
$$S = \{(x_1, \ldots, x_{n+1}) : \sum_i x_i^2 - 1 = 0, x_{n+1} \leq 0\}$$

Since $f : D^n \to D^n$ can be viewed as a map that sends a hemisphere to itself. Since we want everything to map to the southern hemisphere, we can consider f as a map $S \to S$. Now our strategy can be laid out: Using the antipodal map, we can first map N to S and then apply f. Since $N \cap S = \{\vec{x} \in S^n : x_{n+1} = 0\}$, continuity implies that we need to ensure that the points on the southern hemisphere are mapped antipodally in all coordinates except x_{n+1}. Geometrically, this is to account for the fact that the antipodal map is a reflection. Explicitly, we defined our map $g : S^n \to S \subset S^n$ as,

$$g(x_1, \ldots, x_{n+1}) = \begin{cases} f(-x_1, \ldots, -x_{n+1}) & \text{if } x_{n+1} \geq 0 \\ f(-x_1, \ldots, -x_n, x_{n+1}) & \text{if } x_{n+1} \leq 0 \end{cases} \tag{1}$$

It is clear that $\lim_{x_{n+1} \downarrow 0} g = \lim_{x_{n+1} \uparrow 0} g$ so that the g is a continuous extension of f. Moreover g is not surjective as a map $S^n \to S^n$ since points on the northern hemisphere don't have an inverse. Hence $\deg g = 0$ Now as in the one-dimensional case, we consider the deformation retraction $F : I \times S^n \to S^n$,

$$F(t, x) = \frac{(1-t)x - tg(x)}{\|(1-t)x - tg(x)\|_{\mathbb{R}^{n+1}}} \tag{2}$$

If f has no fixed points, then g has no fixed points and $F(t, x)$ is well-defined.[1] However note that F serves as a homotopy between $\mathbb{1}$ and g. But $\deg \mathbb{1} = 1$ while $\deg g = 0$ so that we have a contradiction.

3

Problem. *Let $f : S^n \to S^n$ be a map of degree zero. Show that there exist points $x, y \in S^n$ such that $f(x) = x, f(y) = -y$. Use this to show that if F is a continuous vector field defined on the unit ball D^n in $\mathbb{R}^n$ such that $F(x) \neq 0, \forall x$, then $\exists$ a point on ∂D^n where F points radially outward and another point on ∂D^n where F points radially inward.*

We can effectively use the arguments of Brouwer's Fixed Point Theorem to deduce that f must have a fixed point theorem. If f doesn't have a fixed point then the function

$$F(t, x) = \frac{tx - (1-t)f(x)}{\|tx - (1-t)f(x)\|_{\mathbb{R}^{n+1}}}$$

[1] For a more thorough argument we could show that the denominator never maps to zero. However, this is proved in Theorem 1.1 and the proof doesn't change in high dimensions

is well-defined, continuous and non-singular. However this implies that $f \simeq \mathbb{1}$ even though $\deg f \neq \deg \mathbb{1}$. Hence f must have a fixed point.

One the other hand, if we define a map

$$F(t,x) = \frac{tx + (1-t)f(x)}{\|tx + (1-t)f(x)\|_{\mathbb{R}^{n+1}}}$$

Then if $f(x) \neq -x, \forall x$, this map is well-defined and continuous at $t = \frac{1}{2}$ (again, using the arguments of Theorem 1.1). This gives a homotopy $f \simeq -\mathbb{1}$ even though this implies that $0 = \deg f = \deg -\mathbb{1} = (-1)^{n+1}$, a contradiction. Hence there exists a point y such that $f(y) = -y$.

Define the continuous, unit vector field $G : D^n \to S^{n-1} \subset TD^n \oplus ND^n \cong \mathbb{R}^n$ by,

$$G(x) = \frac{F(x)}{\|F(x)\|_{\mathbb{R}^n}}$$

Note that if G has a pair of inward pointing and outward pointing vectors, then F must have one. On the boundary $\partial D^n \cong S^{n-1}$, we can treat this as a map $\tilde{G} : S^{n-1} \to S^{n-1}$. More precisely, $\tilde{G} = G \circ \iota$, where $\iota : S^{n-1} \hookrightarrow D^n$ is the natural inclusion. Hence $\tilde{G}_* = G_* \iota_*$. But this map factors through $H_n(D^n)$ so $G_* = 0$ and we must have $\tilde{G}_* = 0$. Hence $\deg \tilde{G} = 0$. The conclusion is immediate from the previous results.

4

Problem. *Construct a surjective map S^n of degree zero for each $n \geq 1$*

The strategy here is to construct a map like the one constructed in problem 1 via a deformation and then send the image of the deformation make to a sphere. In the case of S^1 this is quite trivial — let $r : S^1 \to S^1$ be the reflection map that sends the upper arc of S^1 (i.e. $\theta \in [0, \pi)$) to $[\pi, 2\pi)$. Explicitly, we have:

$$r(\theta) = \begin{cases} -\theta & \text{if } \theta \in [0, \pi) \\ \theta & \text{if } \theta \in [\pi, 2\pi) \end{cases}$$

Now if d is the doubling map $\theta \mapsto 2\theta$, then the combination $d \circ r : S^1 \to S^1$ is surjective since $d([\pi, 2\pi)) = S^1$.

Recall from Chapter 0 (page 10) that $S^m \wedge S^n = S^{m+n}$ where $\wedge$ is the smash product. This is constructed by $(S^n \times S^m)/(S^n \vee S^m)$ so that we can extend a map $f : S^1 \to S^1$ to a map $g = q \circ (f \times f) \in \mathsf{End}(S^1 \wedge S^1) \cong \mathsf{End}(S^2)$ where $q : S^n \times S^m \to S^n \wedge S^m$ is the quotient map. Since g is constructed as the composition of surjective maps, g is surjective. Since $S^1 \wedge S^1 \cong \Sigma(S^1) \simeq S(S^1)$ (see page 10), where $\Sigma(S^1)$ is the reduced suspension, the degree of g as a map $S^2 \to S^2$ is the same as the degree of f via Proposition 2.33. Hence g is a degree zero, surjective map on S^2.

We can proceed by induction since $S^m \cong \wedge_{i=1}^m S^1$. The proof is precisely the same, since $S^1 \wedge X \cong \Sigma X$ for any space X.

6

Problem. *Show that every map $S^n \to S^n$ can be homotoped to have a fixed point if $n > 0$*

There are two ways to do this. The first is far more geometric, but relies on Cartan's notion of homogeneous spaces. The spheres S^n admit a transitive group action by the spin groups $\mathsf{Spin}(n)$ that are the universal covers of $\mathsf{SO}(n)$. Since $\mathsf{Spin}(n)$ are simply connected and path connected, the action of an element $g \in \mathsf{Spin}(n)$ is homotopic to the action of identity $\mathbb{1}_{\mathsf{Spin}(n)} \in \mathsf{Spin}(n)$. Hence if we pick a point $x \in S^n$ and it's image $f(x) \in S^n$, we can construct a one-parameter semi-group $g : [0,1] \to \mathsf{Spin}(n)$ such that $g(1) \cdot f(x) = x, g(0) \cdot f(x) = \mathbb{1}_{\mathsf{Spin}(n)} \cdot f(x) = f(x)$. This gives a homotopy to a map that has a fixed point and if f is smooth, this homotopy is also smooth. Moreover, note that this is not necessarily a homeomorphism, since we can only define the action as a one-parameter semi-group as opposed to closed subgroup of $\mathsf{Spin}(n)$.

The purely topological way to answer this question is to us a fact stated at the beginning of §2.2:

$$\deg(f) = \deg(g) \iff f \simeq g$$

We can now proceed by induction. The base case has already been proven in example 2.32, where the maps fixed $f(0) = 0$. Now suppose that $f : S^n \to S^n$ has $f(x_0) = x_0$ and $\deg f = k$. The inductive step follows from Proposition 2.33, so that since $\deg Sf = \deg f$ and subsquently $Sf \simeq f$ and as $S(S^n) \cong S^{n+1}$ (via the smash product argument of earlier), this implies that $Sf|_{S^n} = f$ so $Sf(x_0) = x_0$.

7

Problem. *For an invertible linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ show that the induced map on $H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \approx \tilde{H}_{n-1}(\mathbb{R}^n - \{0\}) \approx \mathbb{Z}$ is $\mathbb{1}$ or $-\mathbb{1}$ according to whether to determinant of f is positive or negative.*

The main fact from linear algebra that I will use here is that linear transformations of positive determinant sends an ordered, oriented basis to an ordered, oriented basis of the same orientation and ordering. Now that an invertible linear transformation is a homeomorphism so in particular $\deg f = \pm 1$. Now let's consider the induced map $f_* \in \mathsf{End}(H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}))$. Without the loss of generality, we can assume that we are dealing with simplical homology. Each point of a chain $\sigma : \Delta^n \to \mathbb{R}^n$ admits a local frame on each face in Δ^n. Under a linear transformation of positive determinant, the orientation of each of these frames has been preserved (i.e. none of the basis vectors changed sign). Recall that one can describe a general Δ-simplex in terms of barycentric coordinates $(t_1, \dots, t_n)$ that effectively describe how a linear transformation takes the standard n-simplex to Δ. Orientation preservation is equivalent to saying that the signs of the $(t_1, \dots t_n)$ were preserved under the linear transformation. This implies that we are **not** changing the generator of H_n^{Δ} since a continuous deformation from barycentric coordinates $(t_1, \dots, t_n)$ to $(t'_1, \dots, t'_n)$ that changes sign in a coordinate t_i will have to pass through $t_i = 0$ in which case the n-simplex degenerates to an $n-1$-simplex. Note that application of the quotient map to pass from chain groups to homology groups will not change orientation since all chains in an equivalence class are related by boundaries and/or barycentric coordinate transformations. Hence under an orientation-preserving, $\det f > 0$ transformation $\mathbb{1} \mapsto \mathbb{1}$ (in homology).

On the other hand, applying an orientation-reversing or negative determinant transformation will be equivalent to negating one of the barycentric coordinates. Hence any chain will be taken to a different equivalence class under f_* and since $\deg f = \pm 1$, this implies that orientation reversing transformations map $\mathbb{1} \mapsto -\mathbb{1}$ in homology.

8

Problem. *A polynomial $f(z)$ with complex coefficients, viewed as a map $\mathbb{C} \to \mathbb{C}$ can always be extended to a continuous map of one-point compactifications $\hat{f} : S^2 \to S^2$. Show that the degree of $\hat{f}$ equals the degree of f as a polynomial. Show also that the local degree of $\hat{f}$ at a root of f is the multiplicity of the root.*

First, consider $g = f|_{S^1}$ where $S^1 \hookrightarrow \mathbb{C}$. Now note that the winding number of a polynomial is a topological invariant that is equivalent to it's (topological) degree[2]. In this case, the winding number of a degree n polynomial is n, so $\deg g = n$ and subsequently since f is an entire function $\deg f = n$. Now note that $\deg Sf = \deg f$ and since $S(S^1) \cong S^2$, we have a map $Sf : S^2 \to S^2$ that is equal to the one-point compactification map. Hence $\deg f = n$

Suppose that the polynomial has roots $r_1, \dots, r_k \in \mathbb{C}$ and via the fundamental theorem of algebra $f(z) = \prod_{i=1}^n (z - r_i)^{\alpha_i}$, where α_i is the multiplicity. Now let D_i be centered at r_i. By a homeomorphism we can ensure that $D_i \cap D_j = \emptyset, \forall i, j$. In each of the punctured disks $D_i - \{r_i\}$, we can construct a closed, but not exact 1-form, $d\theta_i$. In fact, this 1-form is dual (via the de Rham pairing) to the homology generator of each $H_1(D_i - \{r_i\})$ since $(D_i - \{r_i\}) \simeq S^1$. We want to find the relative homology groups $\tilde{H}_k(D_i, D_i - \{r_i\})$. Since $\tilde{H}_k(D_i) = 0, \forall k$, the long exact sequence for relative homology gives $\tilde{H}_2(D_i, D_i - \{r_i\}) \cong \tilde{H}_1(D_i - \{r_i\}) \cong \mathbb{Z}$. Hence, the local degree is well-defined and can be found using winding number integral.

Now note that $\int_{D_i - \{r_i\}} \frac{(z-r_i)^{\alpha_i}}{f(z)} d\theta_i = 0$ since $\frac{(z-r_i)^{\alpha_i}}{f(z)}$ is holomorphic on $D_i - \{r_i\}$. To compute the winding number we have (via the argument principle),

$$\int_{D_i - \{r_i\}} \frac{f'(z)}{f(z)} d\theta_i = 2\pi i \alpha_i$$

Hence the winding number of $f(z)$ is α_i

[2]The reason for this is because the 1-form $d\theta$ is not exact so $\frac{1}{2\pi} \int f(\theta) d\theta$ corresponds (via the de Rham pairing) to a non-trivial element of $H^1(S^1; \mathbb{Z})$. Since S^1 is paracompact, $H^1(S^1; \mathbb{Z}) \cong H_1(S^1; \mathbb{Z})$