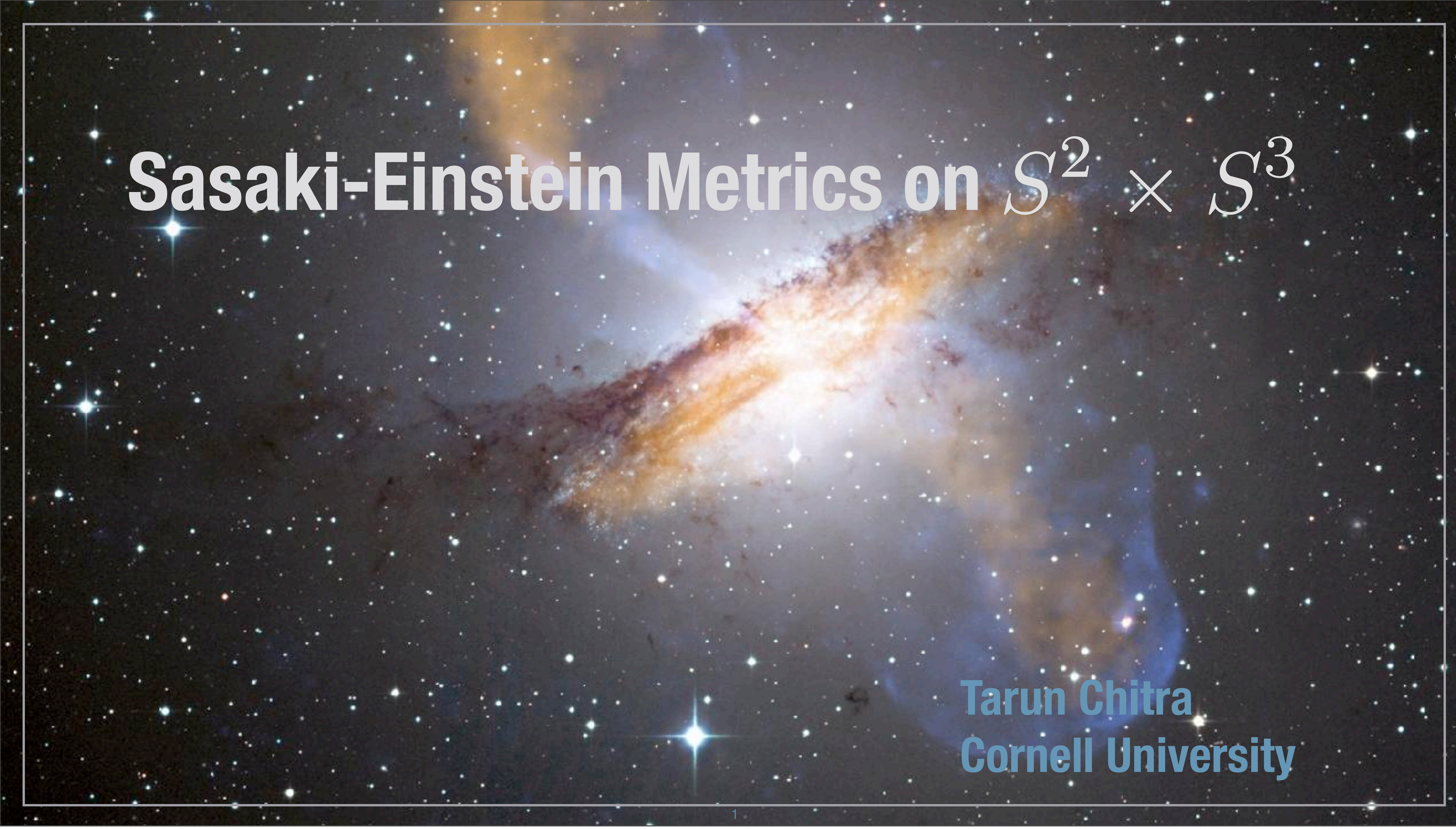

Sasaki-Einstein Metrics on $S^2 \times S^3$

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Mother Nature

- Nature has four forces
  - Familar: Gravity, Electromagnetism (EM)
  - Not-so-familiar: Weak, Strong
- General Relativity explains gravity at big scales and can be adapted to handle the EM, Weak and Strong forces.
- Quantum Mechanics explains EM, weak and strong forces at small length scales
- Fundamental Problem (F.P.): What is a quantum mechanical explanation of gravity? Why do black holes have thermodynamic properties?

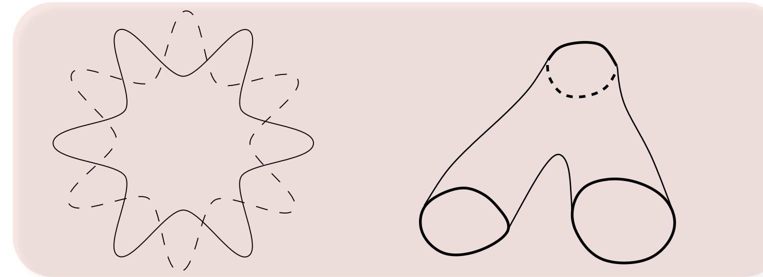
1. Nature has four forces, Gravity, Electromagnetism, the Weak Force and the Strong Force. In the modern world, we have a lot of experience with gravity and electromagnetism and our a high school education ensures that we are aware of these forces. These forces have been well-studied and we have pretty much uncovered how these forces work on a large scale (General Relativity) and on small scale (Quantum Mechanics). Moreover in the past eighty years, we have discovered two other microscopic forces that appear just as fundamental as Gravity and EM — the weak and strong forces. Without going into details, the weak force represents the radiation one encounters while getting an X-Ray or CT scan, while the strong (nuclear) force is what keeps the subatomic particles (e^{-}, \gamma) we know and love together.

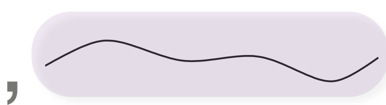
2. The weak and strong force are severely limited in their range and generally only play a role in physics at a very small, internuclear level. However, they do play a role in “big” objects when one considers neutron stars and nuclear fusion in the sun. However, one is more interested in how EM behaves when put in the framework of GR since EM effects are pretty common (e.g. Cosmic Microwave Background radiation). On the other hand, one uses quantum mechanics to study interesting phenomena such as paramagnetism and diamagnetism that cannot be explained solely with Newton’s Laws. Yet an explanation of gravity on a small length scale, such as in quantum mechanics has evaded physicists for years. You might ask why would anyone care? Answer: Black Holes.

As the point particle evolves, it traces out a path, which is a continuous mapping of the interval into a space

Strings

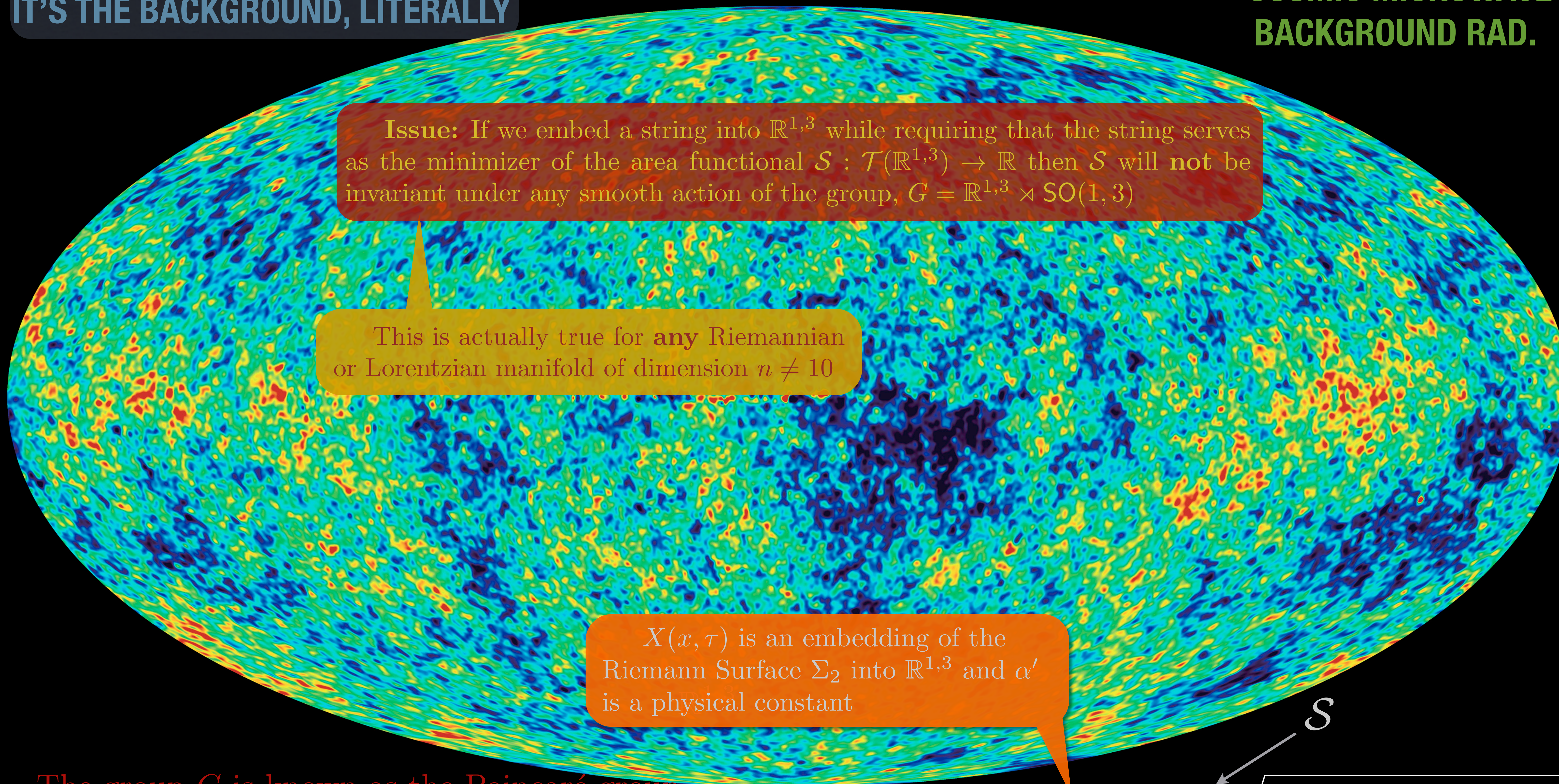
As the string evolves in time within an n-manifold, it traces out a Riemann Surface



- One solution to the F.P. is to extend the point-like object of a field theory to a string, which should be viewed as a 1-dimensional “object”.
- There are only two 1-dimensional manifolds: $I = [0, 1],\ S^1$
- The first case is the open string,  , while the second case is the closed string from the illustrations at the top
- **Question:** Strings are a simple idea, so shouldn’t this *simplify* the hefty equations of Quantum Field Theory and General Relativity?
Answer: Probably, but not not without further geometric implications

IT'S THE BACKGROUND, LITERALLY

COSMIC MICROWAVE BACKGROUND RAD.



Issue: If we embed a string into $\mathbb{R}^{1,3}$ while requiring that the string serves as the minimizer of the area functional $\mathcal{S} : \mathcal{T}(\mathbb{R}^{1,3}) \to \mathbb{R}$ then $\mathcal{S}$ will **not** be invariant under any smooth action of the group, $G = \mathbb{R}^{1,3} \rtimes \mathsf{SO}(1,3)$

This is actually true for **any** Riemannian or Lorentzian manifold of dimension $n \neq 10$

$X(x,\tau)$ is an embedding of the Riemann Surface Σ_2 into $\mathbb{R}^{1,3}$ and α' is a physical constant

$\mathcal{S}$

$$\mathcal{S}(g) = -\frac{1}{2\pi\alpha'}\int_{\Sigma_2}\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}$$

The group G is known as the Poincaré group and it represents all rotations and translations in $\mathbb{R}^{1,3}$

Backgrounds

- The change to a string forces us to consider only 10-dimensional Riemannian/Lorentzian Manifolds — This has huge consequences
- In order to preserve all known physics, one initially considers the 10-manifold for the background to be of the form:

$$M = \mathbb{R}^{1,3} \times X_6$$

where X_6 is a 6-dimensional Kähler Manifold

- Why a Kähler Manifold?
  - Automatically satisfies the integrability conditions so that ODE existence theorems can be used
  - Integrability is easy: The Newlander-Nirenberg Theorem, Nijenhuis Tensor

Kähler Manifolds

If this definition doesn't feel motivated, you're not alone!

Definition:
A Complex, Riemannian Manifold (M, g) is said to be *Almost Hermitian* if $g_p(X, Y) = \overline{g_p(Y, X)}, \forall p \in M, \forall X, Y \in T_pM^{\mathbb{C}}$. An integrable, almost Hermitian manifold is said to be *Kähler* if the 2-form $\omega(X, Y) := g(X, JY)$ is closed.

J is the Complex Structure which heuristically can be described by the phrase: "Separates 'holomorphic' tangent vectors from 'antiholomorphic' tangent vectors"

- The main advantages that Kähler Manifolds have are that they are Riemannian, Complex (Integrable) and Symplectic. In other words, the structure group for the tangent bundle is simply:

$$G = \underbrace{\mathsf{O}(2n)}_{\text{Riemannn}} \cap \underbrace{\mathsf{GL}(n, \mathbb{C})}_{\text{Complex}} \cap \underbrace{Sp(2n)}_{\text{Symplectic}} \cong \underbrace{U(n)}_{\text{Kähler}}$$

Kähler Manifolds: Geometry

- Let's consider how such a Kähler manifold 'looks' geometrically.
- This is more easily seen in local coordinates
- Locally, we can write the Hermitian metric as $g \to g_{\alpha\bar{\beta}}\, dz^{\alpha} \otimes d\bar{z}^{\beta}$ (Einstein Summation Convention implied) and the Kähler form takes the form $h \to \frac{i}{2} h_{\alpha\beta}\, dz^{\alpha} \wedge d\bar{z}^{\beta}$
- The Hermiticity condition is quite strong; in fact, we can prove that the Christoffel Symbols for the Levi-Civita Connection are simply,

$$\Gamma^{n}_{ij} = \sum_{k} g^{n\bar{k}} \frac{\partial g_{j\bar{k}}}{\partial z_i}$$

Note that this reduces the number of derivatives of g to be computed

Kähler Geometry

- Note that this form of the metric separates the holomorphic and anti-holomorphic parts of the metric, implying (at least locally) that the connection splits as, $\nabla = \nabla^{1,0} + \nabla^{0,1}$. Heuristically, this says that parallel transport along a loop preserves "type of vector." In other words,

$$\mathrm{Hol}_g = \mathsf{U}(n)$$

- This gives a Ricci tensor of the form,

$$\mathrm{Ric}_\omega = R_{i\bar{j}} dz^i \otimes d\bar{z}^j = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \ln g \, dz^i \otimes d\bar{z}^j$$

which reduces a computation of the Ricci tensor to a Dirichlet problem for compact manifold with boundary.

- Given this simple form, one may wonder how to solve simplest Laplacian problem for this metric, namely $\mathrm{Ric}_g = 0$

Symplectic Manifolds

- Phase Spaces in physics can be described in terms of the *cotangent bundle* of a manifold
- Fundamental Assumption of mechanics: The change in momentum *and* the change in position can be measured simultaneously at arbitrary precision, or in other words (implictly used: Darboux's Theorem)

$$dp \wedge dq = 0$$

- The *symplectic form* is a generalization of this — it is a non-degenerate, closed, 2-form.
- The Kähler Form *is* a symplectic form, which means that a Kähler Manifold is equipped to handle classical mechanics

Calabi-Yau Manifold

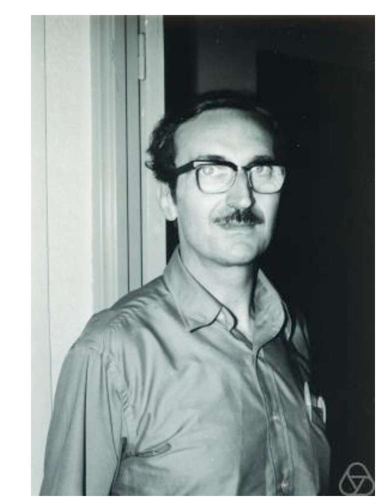
- Remember that we had a Dirichlet problem for the Ricci tensor of a Kähler Manifold?
- It turns out that solving $\mathrm{Ric}_g = 0$ is too strong of a condition; instead for physics, one is interested in solving $\mathrm{Tr}(\mathrm{Ric}_g) = 0$ [Intrinsic Curvature, Ricci Scalar]
- However, it turns out that there are topological obstructions to solving this equation, namely the admission of a non-trivial line bundle makes it impossible to solve the above equation

Calabi-Yau Manifolds: Brief History

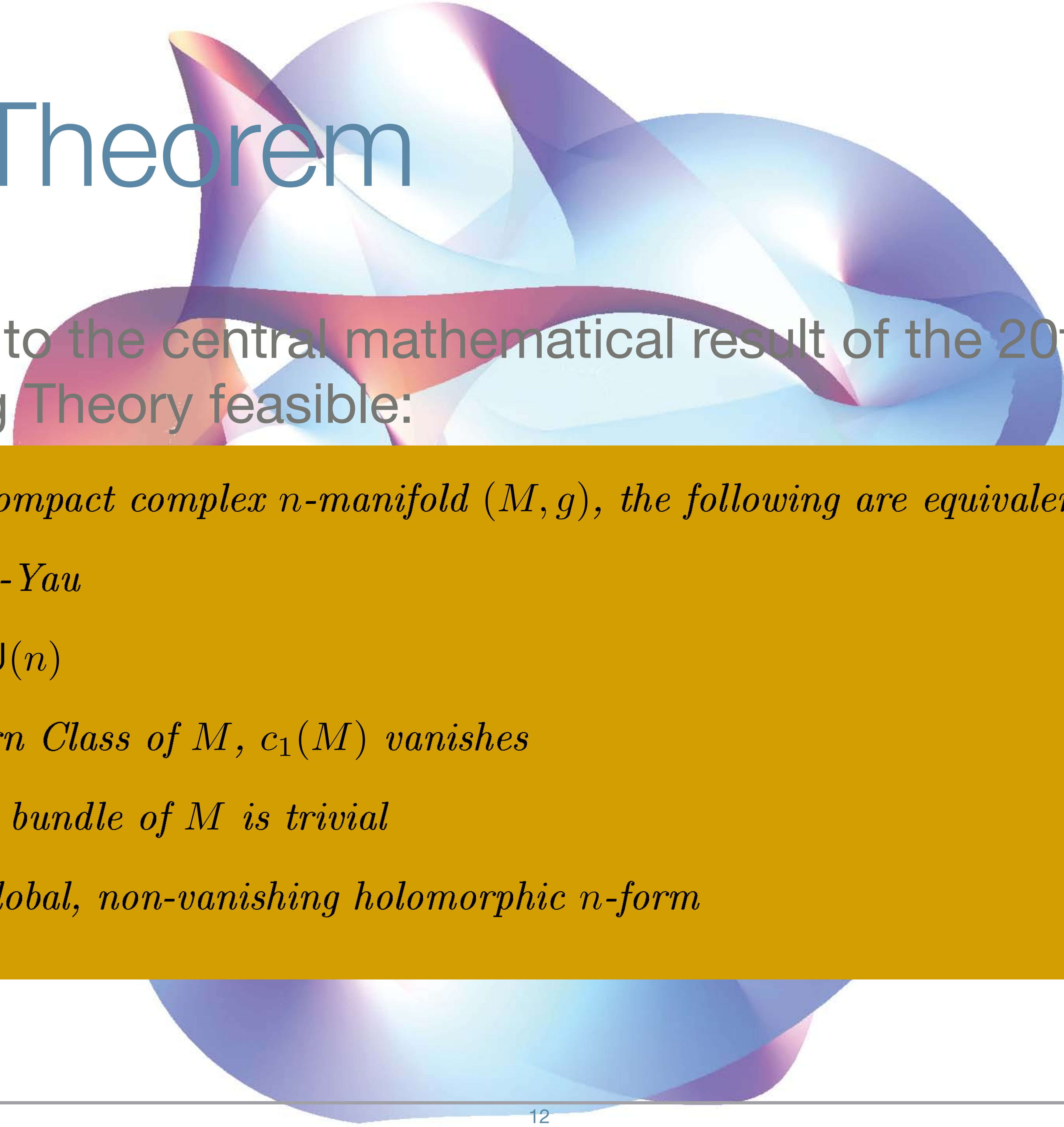
- Historically, Kähler Manifolds became important when Chern showed that he could classify line bundles of Kähler Manifolds via a formula involving the Kähler Form

- Chern generalized this to a purely topological statement using *characteristic classes* in the 1946

- In 1956, Calabi conjectured that this constraint implies that we have a solution for the Ricci Scalar problem and 1970s, evidence had stacked up for this conjecture and Shiing-Tung Yau proved this in 1978



Yau's Theorem



- We've come to the central mathematical result of the 20th century that makes String Theory feasible:

Theorem. *For a compact complex n-manifold (M, g), the following are equivalent:*

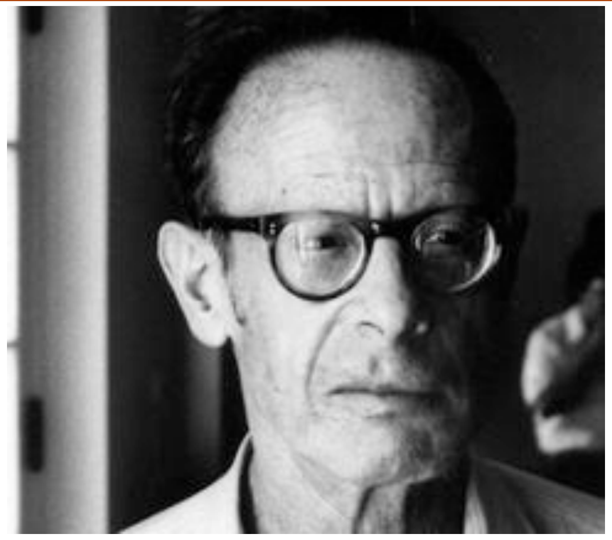
- *M is a Calabi-Yau*
- *$Hol_g(M) \subset \mathsf{SU}(n)$*
- *The first Chern Class of M, $c_1(M)$ vanishes*
- *The canonical bundle of M is trivial*
- *M admits a global, non-vanishing holomorphic n-form*

Calabi-Yau Examples

The simplest 3-dimensional (complex) Calabi-Yau Manifold is defined by the zero locus,

$$Z = \{[Z_1, Z_2, Z_3, Z_4, Z_5] \in \mathbb{C}P^4 : Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 + Z_5^2 = 0\}$$

Yau's Theorem effectively says that the only line bundle over this space is the canonical bundle



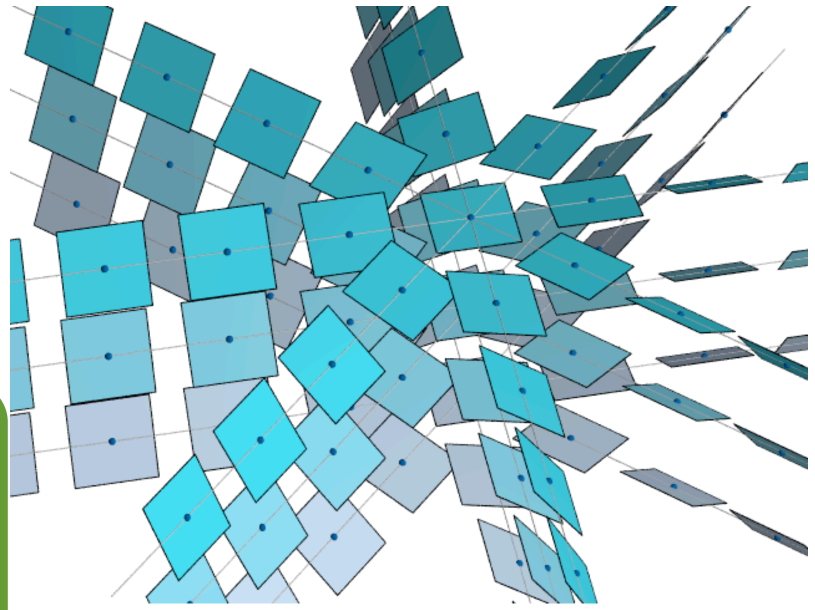
Another common example are the K3 surfaces, which are in general not algebraic, so we cannot write them as zero loci in any projective space

Problem: All of the known examples of algebraic Calabi-Yau Manifolds are non-compact. Since compact Calabi-Yau Manifolds are of interest in physics, one would like to find an explicit metric on a compact Calabi-Yau manifold — This has not been done yet!

Sasakian Manifolds

- Sasakian Manifolds can be looked at as the “odd-dimensional cousin” of Calabi-Yau Manifolds
- They have *contact* and *CR structures* which complement the symplectic and complex structures on a Calabi-Yau Manifold
- There is a fundamental relationship between a Sasakian Manifold (M, g) and it’s metric cone, $(C(M), \tilde{g}), \tilde{g} = dr^2 + r^2 g$
- This is often taken as the definition in physics, albeit without considering the limit $r \uparrow \infty$

Contact Structures



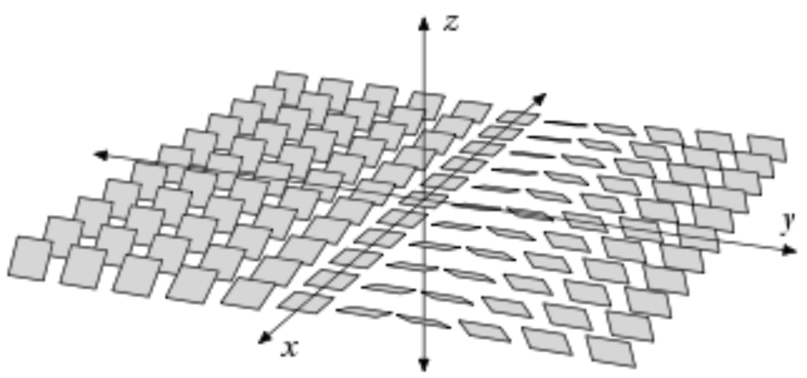
Definition. A *contact structure* on a Riemannian Manifold M of dimension $2n+1$ is a choice of smooth $2n$-dimensional tangent distribution with a specific integrability condition.

On a local trivialization U_α of TM, we can define the distribution $D \subset T_pM, \forall p \in U_\alpha$ by defining a 1-form α such that $\alpha|_D = 0$. In the local setting, this integrability condition becomes,

$$\alpha \wedge \overbrace{d\alpha \wedge \cdots \wedge d\alpha}^{n} \neq 0$$

The standard example of a contact form on $\mathbb{R}^3$ is: $\theta = \frac{1}{2}dz + \sum_i y_i dx_i$
The zero set of this 1-form is depicted below

Fact 1. The metric cone of a contact manifold is symplectic
Fact 2. Contact manifolds have a canonical, unit norm vector field $X \in \Gamma(TM)$ such that $\omega(X) = 1, d\omega(X,Y) = 0\, \forall Y \in \Gamma(TM)\, \forall \omega \in TM$



Sasakian Manifold: Definition

Theorem. Suppose (M, g) is a $2n+1$-dimensional Riemannian Manifold The following are equivalent:

- M is a Sasakian Manifold
- $\exists$ a global, unitary Killing vector field ξ such that the Ricci tensor satisfies the following equation

$$R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi$$

  where η is 1-form dual to ξ via the tangent-cotangent isomorphism
- The metric cone $C(M)$ is Kähler

Thus a Sasaki-Einstein Manifold is both Sasakian and Einstein

Recall that an Einstein Manifold is a manifold whose curvature tensor is proportional to its metric tensor, $\mathrm{Ric}_g(X,Y) = \kappa g(X,Y)$

Sasakian Isometries

- The definition implies that there exists a global, unitary Killing Vector that is a contact 1-form
- This means that the manifold has a U(1) isometry and admits a smooth, free and proper, U(1) action.
- One can classify Sasaki-Einstein Manifolds by the quotient of this U(1) action, as the quotient will be Kähler [Note: 4-dim. Kähler Manifolds have been completely classified by Shiing-Tung Yau and Gang Tian]

The Reeb Foliation

- The orbits of the flows associated to the Reeb Vector Field are U(1) and foliate the manifold with 1-dimensional spaces. We consider the Sasakian Manifold M and the quotient by these orbits, N
- One classifies these foliations into three classes
  - Regular Foliation: An orbit is homotopic to S^1 and the fiber has a "winding number" of one
  - Quasi-Regular Foliation: An orbit is also homotopic to S^1 and the fiber has a "winding number" of k
  - Irregular Foliation: The orbit does not close

Viewpoint Differential Geometry in 1995: Irregular Sasaki-Einstein Manifolds do NOT exist

The AdS/CFT Conjecture

Theorem. *If* $\dim M = 4$ *and the boundary metric* γ *is of class*[5] $C^{7,\alpha}$. *Then the pair* $(\gamma, g_{(3)})$ *on* ∂M *uniquely determined an Asymptotically Hyperbolic Einstein metric up to local isometry. This means that if* g^1, g^2 *are two AH Einstein metrics on manifolds* M_1, M_2 *with* $\partial M = \partial M_1 = \partial M_2$ *such that with respect to the aforementioned compactifications* $(\bar{M}_1, \bar{g}_1), (\bar{M}_2, \bar{g}_2)$, *we have:*

$$\gamma^1 = \gamma^2 \quad \textit{and} \quad g^1_{(3)} = g^2_{(3)}$$

then g^1, g^2 *are locally isometric and* M_1, M_2 *have diffeomorphic universal covers.*

Let $(\bar{M}, \bar{g})$ be an $n+1$-dimensional Riemannian Manifold with boundary and let $M = \mathring{\bar{M}}, \partial M = \partial \bar{M}$. We say that g is *asymptotically hyperbolic* if $\bar{g}|_{\partial M}$ is hyperbolic and Einstein.

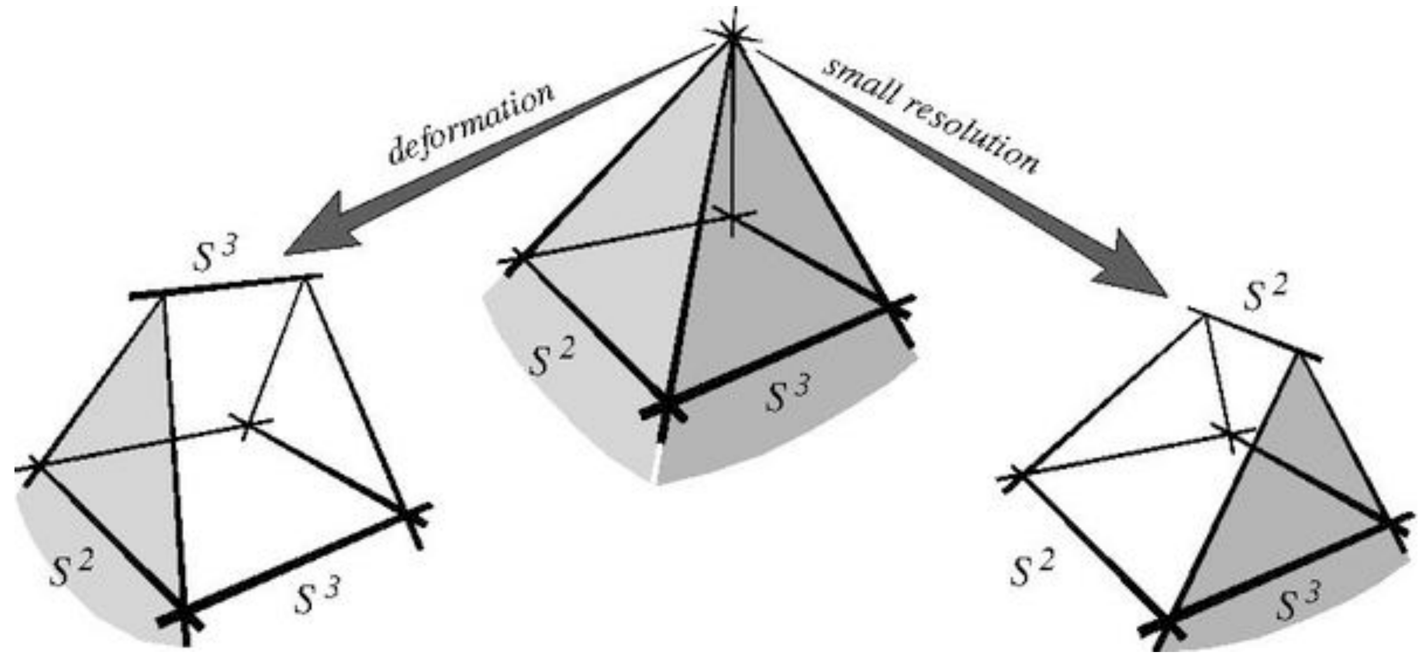
Now using the Gauss Lemma, we can show that the metric on $\bar{M}$ can be written as $\bar{g} = dt^2 + g_t$, where g_t is a family of metrics on the hypersurfaces $t = t'$. We define the Fefferman-Graham Expansion of (M, g) by

$$g_t = g_0 + t g_{(1)} + t^2 g_{(2)} + \ldots + t^n g_{(n)} + O(t^{n+\alpha})$$

Gauge/Gravity Duality

- Physically, Gauge/Gravity duality is a generalization of AdS/CFT that generalizes the correspondence from a metric theorem to a topological theorem

- Mathematically, Gauge/Gravity duality is a series of *blow-ups* and *blow-downs* that give a method for resolving a singularity in a compact, 4-manifold with boundary of General Relativity to a vector bundle over the boundary. For example:



$Y^{p,q}$

$$g(\partial_t,\partial_\phi,\partial_\psi,\partial_r,\partial_\theta) = -\frac{\Delta}{\rho^2}\left(dt - \frac{a\sin\theta}{\Xi_a}d\phi - \frac{b\cos^2\theta}{\Xi_b}\right)^2 + \frac{\Delta_\theta\sin^2\theta}{\rho^2}\left(a\,dt - \frac{(r^2+a^2)}{\Xi_a}d\phi\right)^2 + \frac{\Delta_\theta\cos^2\theta}{\rho^2}\left(b\,dt - \frac{r^2+b^2}{\Xi_b}d\psi\right)^2$$
$$+\frac{\rho^2}{\Delta}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{(1+r^2l^{-2})}{r^2\rho^2}\left(ab\,dt - \frac{b(r^2+a^2)\sin^2\theta}{\Xi_a}d\phi - \frac{a(r^2+b^2)\cos^2\theta}{\Xi_b}d\psi\right)^2 \tag{7}$$

$$\Delta = \tfrac{1}{r^2}(r^2+a^2)(r^2+b^2)(1+r^2l^{-2}) - 2M$$

$$\Delta_\theta = (1 - a^2l^{-2}\cos^2\theta - b^2l^{-2}\sin^2\theta$$

$$\rho^2 = (r^2 + a^2\cos^2\theta + b^2\sin^2\theta)$$

$$\Xi_a = (1 - a^2l^{-2})$$

$$\Xi_b = (1 - b^2l^{-2})$$

Physical Implications

- Superconductors
- Inflation

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  - Convinced me that Beamer is unsuitable for presentations!