Analytic Capacity

Tarun Chitra

Outline

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Problem

Definition

A set K is said to be *removable* if for whenever $K \subset \Omega \subset \mathbb{C}$, where $\Omega \subset \mathbb{C}$ is open, every function $f : \Omega \backslash K \to \mathbb{C}$ has an analytic extension to all of Ω.

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Can we determine which sets in the complex plane are removable? What geometric and analytic properties about these sets are important? Can we classify a measurement of "how removable" a set is?

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Early History

- We are familiar with Riemann's famous theorem on removable singularities from Stein & Shakarchi:

Theorem

(Riemann) Suppose that f is holomorphic in an open set Ω except possibly at a point $z_0 \in \Omega$. If f is bounded on $\Omega - \{z_0\}$ then z_0 is a removable singularity

- French mathematician and politican Paul Painlevé wondered if it was possible to further characterize compact removable sets in $\mathbb{C}$. Lars Ahlfors, in 1942, restated what is contemporarily known as the *Painlevé Problem* in the following succinct form:

Given a compact set $E \subset \mathbb{C}$, when does there exist a non-constant bounded analytic function $f(z)$ on $\mathbb{C} \backslash E$?

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Painlevé's Theorem

Theorem

(Painlevé) Assume that for all $\varepsilon > 0$, the compact set $E \subseteq \mathbb{C}$ can be covered by a collection of discs whose radii does not exceed ε. Then the set of bounded analytic functions on $\mathbb{C} \backslash E$ consists only of constants[a]

[a] Note that the set of bounded analytic functions on a set Ω is denoted $H^\infty(\Omega)$, with the notation representing the H^∞ Hardy Space

Proof.

For each $\varepsilon > 0$, cover E by a collection of discs U_i such that $\sum_i r_i < \varepsilon$ (where r_i is the radius of the disc U_i). Now let $D_\varepsilon = \bigcup_i U_i$ and let $\Gamma_\varepsilon = \partial D_\varepsilon$. Next, setup the following Cauchy Integral

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} \frac{f(\zeta)}{z - \zeta} d\zeta \quad z \notin \overline{D}_\varepsilon$$

for some $z \in \mathbb{C} \backslash E$ and $f \in H^\infty(\mathbb{C} \backslash E)$ with $f(\infty) = 0$. Then $|f(z)| \leq \frac{\varepsilon \cdot \sup_{\Gamma_\varepsilon}(|f|)}{2\pi \cdot d(z, \Gamma_\varepsilon)}$. As $\varepsilon \searrow 0$, $|f(z)| = 0$. □

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   - Removable Sets
   - Analytic Condition for Removability
2. Preliminaries for Modern Results
   - Hausdorff Measure and Hausdorff Measure
   - Rectifiability
3. Modern Results
   - Hausdorff Measure and Analytic Capacity
   - Garnett's Counter-Example
   - Denjoy and Vitushkin Conjectures

What is Analytic Capacity?

- Ahlfor's came up with the notion of the *Analytic Capacity* of a set E, $\gamma(E)$ defined by:

$$\gamma(E) = \sup\{|f'(\infty)| : f : \mathbb{C}\backslash E \to \mathbb{C} \text{is holomorphic}, ||f||_\infty \leq 1\}$$

where $f'(\infty)$ is calculated relative to the local coordinate $z = \frac{1}{\zeta}$ on the Riemann Sphere as :

$$f'(\infty) = \lim_{\zeta\to 0} f'(\frac{1}{\zeta}) = \lim_{\zeta\to 0} \frac{f(\frac{1}{\zeta}) - f(\infty)}{\zeta - \frac{1}{\zeta}} = \lim_{z\to\infty} z(f(z) - f(\infty))$$

and $f(\infty) = \lim_{z\to\infty} f(z)$. Note that in general $\lim_{|z|\to\infty} f'(z) \neq f'(\infty)$.

- Now suppose we look at the Möbius Transformation

$$g(z) = \frac{f(z) - f(\infty)}{1 - \overline{f(\infty)}f(z)}$$

which is in $H^\infty(\mathbb{C}\backslash E)$ as $||f|| \leq 1$. Therefore, we need only consider functions with $f(\infty) = 0$ as $g(\infty) = 0$. If not, then $g'(\infty)$ will always tend to $-\infty$ regardless of $f'(\infty)$.

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Properties of γ

Let's first establish some properties about γ:

- If $f(\infty) = 0$ then γ has an invariance property: $\gamma(aE + b) = |a|\gamma(E)$ which comes from the fact that $\lim_{z\to\infty} zf(az + b) = af'(\infty)$
- γ is monotone: If $E \subset F$ then $\gamma(E) \leq \gamma(F)$
- An important proposition:

Proposition

Assume that E is connected but not a point. Let g be the conformal map of Ω onto the unit disc satisfying $g(\infty) = 0$, $g'(\infty) > 0$. Then $\gamma(E) = |g'(\infty)|$

Proof.

Clearly $|g'(\infty)| \leq \gamma(E)$ by definition of γ. Let f be any other map that is in $H^\infty(\mathbb{C}\backslash E)$ and satisfies $||f||_\infty \leq 1$ and $f(\infty) = 0$. Apply Schwarz's Lemma to $F := f \circ g^{-1}$ to get $|f'(\infty)| \leq |g'(\infty)|$ for all such f so that $\gamma(E) \leq |g'(\infty)|$ $\square$

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$\square$

Example 1: $I \subset \mathbb{R}$

Example

Let $E = [-2, 2]$. Let $g(z) = z + \frac{1}{z}$; note that g is a conformal map that takes the unit circle to E [a]. Now note that $\gamma(E) = g^{-1'}(\infty)$ so that the inverse function theorem gives:

$$\gamma(E) \quad = \quad \frac{1}{g'(g(\infty))} = \frac{1}{1 - \frac{1}{\infty}} = 1$$

Now let $I = [a, b] \subset \mathbb{R}$ so that

$$\gamma(I) \quad = \quad \left(\frac{b-a}{4}[-2, 2] + \frac{a+b}{2} \right) = \frac{b-a}{4}\gamma([-2, 2]) = \frac{1}{4}m(I)$$

Note that with some further effort it can be shown that for all $E \subset \mathscr{B}(\mathbb{R})$ that $\gamma(E) = \frac{1}{4}m(E)$.

[a] see Stein & Shakarchi, Ch. 8.1, Example 5

Example 2: A disc

Example

A disc $D(z_0, r)$ for $z_0 \in \mathbb{C}$, $r > 0$
Let $D = \{z : |z - z_0| \leq r\}$ so that $E^c = \overline{\mathbb{C}} \backslash E = \{z : |z - z_0| \geq r\}$. Then we can map $E^C \mapsto \mathbb{D}$ using the the conformal map $z \mapsto \frac{r}{z - z_0}$. If $g(z) = \frac{r}{z - z_0}$ then using the local coordinate $z = \frac{1}{\xi}$, we have $g(\frac{1}{\xi}) = \frac{r}{\frac{1}{\xi} - z_0} = \frac{r\xi}{1 - z_0\xi}$ and differentiating yields

$$g'(\frac{1}{\xi}) = \frac{r}{1 - z_0\xi} - \frac{r\xi}{(1 - z_0\xi)^2}$$

so that if we send $\xi \to 0$, we have $g'(\infty) = \lim_{\xi \downarrow 0} g'(\frac{1}{\xi}) = r$.

How is Analytic Capacity Related to Removability?

Theorem

Let $E \subset \mathbb{C}$ be a compact set. Then the following assertions are equivalent:

(i) $\gamma(E) = 0$

(ii) Every bounded analytic function $f : \mathbb{C} \backslash E \to \mathbb{C}$ is constant

(iii) E is removable for bounded analytic functions

Proof. It is clear that $(ii) \Rightarrow (i)$. Now suppose for a contradiction that there exists a non-constant bounded analytic function $f : \mathbb{C} \backslash E \to \mathbb{C}$ with $f(\infty) = 0$ and $f(z_0) \neq 0$ for $z_0 \in \mathbb{C} \backslash E$. Now suppose we define

$$g(z) = \begin{cases} \frac{f(z) - f(z_0)}{z - z_0} & \text{if } z \neq z_0 \text{ and } z \in \mathbb{C} \backslash E \\ f'(z_0) & \text{if } z = z_0 \end{cases}$$

Note that $g \in H^\infty(\mathbb{C} \backslash E)$ and that $g'(\infty) = f(z_0) \neq 0$. Therefore $\gamma(E) > 0$ and $\neg(ii) \Rightarrow \neg(i)$. The implication $(iii) \Rightarrow (ii)$ follows from Liouville's theorem.

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Proof of Theorem 2 (Continued)

Now suppose that E satisfies (ii). Then it is claimed that E must be totally disconnected.

- Suppose that E is not totally disconnected; then the Riemann Mapping theorem yields a non-constant, bounded analytic function $f : \mathbb{C} \backslash E_0 \to \mathbb{D}$ for some $E_0 \subseteq E$.

Now let $E \subset U$ for some open set U, f be a bounded analytic function on $U \backslash E$ and fix $z_0 \in U \backslash E$. As E is totally disconnected, there are two curves Γ_1 and Γ_2 such that z_0 is in the domain bounded by Γ_1 and not in the domain bounded by Γ_2. Using the Cauchy Integral Formula we can show that $\int_E dz \frac{f(z)}{z-z_0} = \left(\int_{\Gamma_1} dz + \int_{\Gamma_2} dz \right) \frac{f(z)}{z-z_0}$ is an analytic extension of f to all of U $\blacksquare$

Outline

Hausdorff Measure

Let $E \subset \mathbb{R}^n$, $\mathscr{X}(E)$ be a collection of countable covers in which each disc in the cover has a bounded diameter[1] and set $s > 0$. For $\delta > 0$, define:

$$H^s_\delta(E) = \inf_{\{U_i | i \in I\} \in \mathscr{X}(E)} \left\{ \sum_i (\operatorname{diam} U_i)^s : E \subset \bigcup_i U_i \right\}$$

Now note that $H^s_\delta(E)$ is monotone decreasing in δ; for when $\delta \searrow 0$, there are more covers in $\mathscr{X}(E)$. Therefore the limit as $\delta \searrow 0$ exists.

Definition

The *s-dimensional Hausdorff Measure*, $H^s(E)$ for $E \subset \mathbb{C}$ by:

$$\begin{aligned} H^s(E) &= \lim_{\delta \searrow 0} H^s_\delta(E) \\ &= \sup_{\delta > 0} H^s_\delta(E) \end{aligned}$$

[1] In other words, $\mathscr{X}(E) = \{\cup_i U_i : E \subset \cup_i U_i, U_i \subset \mathbb{R}^n, diam(U_i) \le \delta\}$, where $\operatorname{diam}(U_i) = \sup U_i - \inf U_i$

Properties of Hausdorff Measure

- H^s is a regular Borel measure; that is, H^s can measure all countable unions and intersections of the open and closed sets in $\mathbb{R}^n$ and H^s is both inner regular and outer regular.
- H^s is not a Radon measure (it is not locally finite if $s < n$; this is important in Brownian Motion)
- Relationship between H^1 and m^1 (Lebesgue measure on $\mathbb{R}$):
  - Let $E \subset \mathbb{R}^n$ then $H^n(E) = \frac{\pi^{n/2}}{2^n \Gamma(\frac{d}{2}+1)} m^n(E)$ where m^n is the Lebesgue measure on $\mathbb{R}^n$.
  - $H^1(E) = L^1(E)$
    - $H^n(B(x,r)) = (2r)^n$ for $x \in \mathbb{R}^n$ and $r \in (0,\infty)$
  - The proof of this fact is rather complicated and relies on on the isodiametric inequality: $m^n(A) \leq 2^{-n}\alpha(n)\mathrm{diam}(A)^n \quad \text{for } A \subset \mathbb{R}^n$

Hausdorff Dimension

Definition

The *Hausdorff Dimension* of a set $E \subset \mathbb{R}^n$, $\dim_H E$, is defined by:

$$\begin{aligned} \dim_H E &= \sup_s \{s > 0 : H^s(E) = +\infty\} \\ &= \inf_t \{t > 0 : H^t = 0\} \end{aligned}$$

Note that $\dim_H E$ need not be an integer (unlike the n-dimensional Lebesgue Measure).

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Curves

Definition

A *curve* $\Gamma \in \mathbb{C}$ is a set of the form $\Gamma = \phi([a,b])$ and $\phi : [a,b] \to \mathbb{C}$ is continuous. If ϕ is injective, we say that Γ is a *Jordan curve* and if ϕ is Lipschitz[a] then we say ϕ is a *Lipschitz curve*. Finally, we define the length of Γ, $l(\Gamma)$ by:

$$l(\Gamma) = \sup \sum_{i=1}^{n} |\phi(t_i) - \phi(t_{i-1})|$$

If a curve has $l(\Gamma) < +\infty$, then Γ is said to be *rectifiable*

[a] A map $f : E \subset \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz map if there exists $K > 0$ such that $|f(x) - f(y)| \leq K|x - y|$

Rectifiability

Definition

A set $E \subset \mathbb{C}$ is said to be *1-rectifiable* if there exist Lipschitz maps $f_j : \mathbb{R} \to \mathbb{C}$ such that $H^1(E \setminus \cup_j f_j(\mathbb{R})) = 0$. Less formally, this says that E can be covered by a countable union of Lipschitz curves (up to a set of zero 1-dimensional Hausdorff measure). A set $F \subset \mathbb{C}$ is said to be *purely 1-unrecifiable* if $H^1(E \cap \Gamma) = 0$ for all rectifiable curves $\Gamma \subset \mathbb{C}$.

Frostman's Lemma

Finally, an important result in geometric measure theory that is related to rectifiability and is crucial to demonstrating the relationship between $\dim_H E$ and $\gamma(E)$ is Frostman's Lemma (1935):

Lemma

(Frostman) Let A be a Borel set in $\mathbb{R}^n$. Then $H^s(A) > 0$ if and only if there exists a compactly supported (unsigned) Radon measure μ such that $\operatorname{spt}\mu \subset A$, $0 < \mu(\mathbb{R}^n) < \infty$ and $\mu(B(x,r)) \leq r^s$ for all $x \in \mathbb{R}^n$ and $r > 0$.

Outline:

- $(\Leftarrow)$ falls directly from the definition of H^s
- $(\Rightarrow)$ comes from a construction of a sequence of monotonically-drecreasing Radon measures that depend on $m^n(Q)$ for $n = \lceil s \rceil$ (giving an upper bound for H^s as s need not be in $\mathbb{Z}$ whereas n must be in $\mathbb{Z}$)

Sketch Proof of Frostman's Lemma

(Very) Sketch(y) Proof. ($\Rightarrow$) Let $H^s(A) = b > 0$. Then for some collection of cubes $\{Q_i\}_{i\in I}$ that cover A we have:

$$\sum_i d(Q_i)^s \geq b$$

Now let $m \in \mathbb{N}$ and let $\mathscr{D}_m$ be the family of dyadic cubes of $\mathbb{R}^n$ with side-length 2^{-m} and define a measure μ_m^m on $\mathbb{R}^n$ such that for all $Q \in \mathscr{D}_m$ we have:

$$\begin{aligned} \mu_m^m|_Q &= 2^{-ms} m^n(Q)^{-1} && \text{if } B \cap Q \neq \emptyset \\ \mu_m^m|_Q &= 0 && \text{if } B \cap Q = \emptyset \end{aligned}$$

Let $\mu^m = \mu^m_{m-k_0}$ for k_0 such that $B \subset Q$ for $Q \in \mathscr{D}_{m-k_0}$. Then we have $\mu^m(Q) = 2^{-(m-k_0)s}$ (i.e. there is an ascending chain condition on the set of μ_m^m) an d subsequently:

$$\mu^m(\mathbb{R}^n) = \sum_{i=1}^k \mu^m(Q_i) = n^{-s/2} \sum_{i=1}^k \operatorname{diam}(Q_i)^s \geq n^{-s/2} b$$

Let $\nu^m = \mu^m(\mathbb{R}^n)^{-1}\mu^m$. Then it can be shown that $\nu(B(x,r)) \leq \nu(U) \leq 2^{n+2s} b^{-1} n^{s/2} r^s$.
($\Leftarrow$) Let $\{U_i\}_{i\in I}$ be a countable collection of balls that cover of A with $\operatorname{diam} U_i \leq \delta$ and let $\mu(A) = K$.
Then:

$$K = \mu(A) = \mu\left(\bigcup_{i\in I} U_i\right) \leq \sum_{i\in I} \mu(U_i) \leq \sum_{i\in I} \delta^s \leq H^s_{\delta'}$$

for $\delta' > \delta$. Therefore $H^s(A) = \lim_{\delta' \searrow 0} H^s_{\delta'} = \mu(A) = K > 0$.

Outline

Relationship between Hausdorff Measure and γ

Theorem

Let $E \subset \mathbb{C}$.

(1) If $H^1(E) = 0$, then $\gamma(E) = 0$
(2) If $\dim_H E > 1$, then $\gamma(E) > 0$

Outline:

- For the first part, we simply need to cover E with a cover made up of arbitrarily small disks. Then the Cauchy Integral Theorem will force $f(z)$ to go to zero
- For the second part, we will use Frostman's lemma to guarantee the existence of a non-constant bounded analytic function on E with $|f'(\infty)| > 0$.

Proof of Theorem 3

Proof.

(Sketch) (1) Cover E with a countable cover $\{U_i\}_{i\in I}$. As $H^1(E)=0$, E can be covered by discs $U_i \subset \mathbb{C}$ such that $\sum_i \operatorname{diam}(U_i) < \varepsilon$ for any $\varepsilon > 0$. This means that a circle of diameter ε surrounds E so that we can surround E by a finite collection of C^1 curves Γ_j (i.e. finite subcover) such that $\sum_j l(\Gamma_j) < 2\pi\varepsilon$. Now, choose some z outside of the domain D bounded by the Γ_j, so that the Cauchy Integral Theorem yields

$$f(z) = \frac{1}{2\pi i}\sum_j \int_{\Gamma_j} \frac{f(\xi)}{\xi - z}\, d\xi$$

Subsequently we can bound
$|f'(\infty)| \leq \lim_{z\nearrow\infty} \frac{1}{2\pi}\left|\sum_j \int_{\Gamma_j} \frac{zf(\xi)}{\xi-z} d\xi\right| \leq \lim_{z\nearrow\infty} \varepsilon \sup_{\xi\in\Gamma_j} f(\xi) \to 0$ as $\varepsilon \downarrow 0$ so (1) is proved. $\square$

Proof of Theorem 3 (Continued)

Proof.

(2) Since $\dim_H E > 1$, $H^1(E) > 0$ so that Frostman's Lemma gives a (positive) measure μ that satisfies the previous conditions. Let $f = \frac{1}{z} * \mu = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{1}{z-z'} \mu(z')$; note that f is holomorphic away from E as μ is supported on E. It turns out that $|f(z)|$ is bounded as we can approximate the integral as

$$|f| \leq \frac{1}{2\pi} \int_{|\xi - z| \geq 1} d\mu(\xi) + \sum_j \frac{1}{2\pi} \int_{2^{-j-1} < |\xi - z| \leq 2^{-j}} \frac{1}{|z-\xi|} d\mu(\xi)$$

Note that the bound above is due to the local finiteness of μ. Subsequently note that we have $|f'(\infty)| = \lim_{z \to \infty} \frac{1}{2\pi} \int_{\mathbb{C}} \left| \frac{z}{z-z'} \right| \mu(z') = \frac{\mu(\mathbb{C})}{2\pi} > 0$, which proves (2).

$\square$

Outline

Garnett's Counter-Example
Introduction

- From Theorem 3, one is tempted to conjecture that $H^1(E) = 0 \iff \gamma(E) = 0$. However in 1969, John Garnett constructed an example of a set E such that $H^1(E) > 0$ but $\gamma(E) = 0$ so that $\gamma(E) = 0 \not\Rightarrow H^1(E) = 0$.
- The example he used was the "four-corners" Cantor Set which can be define as follows: Let $E_0 = [0,1] \times i[0,1] \subset \mathbb{C}$, the unit square in the first quadrant of the plane. Then let E_1 be the set of four squares of side length $\frac{1}{4}$ that reside in the four corners of E_0. Similarly, let E_n be the union of 4^n squares located in the four corners of the 4^{n-1} squares of E_{n-1}, with each square in E_n have side length 4^{-n}. Finally, let $E = \cap E_n$ to yield the four-corners Cantor set. In order to simplify notation, label each square in E_n by Q_n^j (for $j = 1...4^n$) so that each $Q_n^j \subset Q_{n-1}^k$ for some k.

Garnett's Counter-Example
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Garnett's Counter-Example
Picture of E_0, E_1, E_2

- Here's a picture of E_0, E_1 and E_2 (from *Analytic Capacity and Measure* by Pajot):

Garnett's Counter-Example

Proof that $H^1(E) \neq 0$

Proof.

Since each Q_n^j is a square, $\operatorname{diam}(Q_n^j) = 4^{-n}\sqrt{2}$. Now fix $\delta > 0$ so that if $4^{-n} < \delta$ then:

$$H_\delta^1(E) \quad \leq \quad \sum_{j=1}^{4^n} \operatorname{diam}(Q_n^j) = \sqrt{2}$$

Therefore $H^1(E) = \lim_{\delta \searrow 0} H_\delta^1(E) \leq \sqrt{2}$ so $H^1(E) < \infty$. In fact we can show that $H^1(E) = \frac{1}{\sqrt{2}}$:

$$H^1(E) \quad \leq \quad \sum_{n=1}^{\infty} 4^{-n}\sqrt{2} = \sum_{n=1}^{\infty} \frac{\sqrt{2}}{2^{2n}} = \frac{\sqrt{2}}{2} \sum_{n=1}^{\infty} \frac{1}{2^{2n-1}} = \frac{1}{\sqrt{2}}$$

$\square$

Garnett's Counter-Example

Preliminaries to show that $\gamma(E) = 0$

- Let $E_{n,j} = E \cap Q_n^j$ (i.e. so that the intersection contains the infinitesmal portion of Q_n^j that is in E).
- Since $E_{n,j}$ is geometrically similar to E, $\gamma(E_{n,j}) = 4^{-n}\gamma(E)$.
- Now let f be such that $||f||_\infty \leq 1$ and $f : \mathbb{C}\backslash E \to \infty$ is holomorphic and $f(\infty) = 0$; suppose that $\gamma(E) > 0$ so that $a = f'(\infty) \in \mathbb{R}^+$. For $z \in \mathbb{C}\backslash E$, let $\Gamma_{n,j}$ be a cycle with winding number one (if possible, let $\Gamma_{n,j}$ be a circle) about $E_{n,j}$ (while having winding number 0 about $E\backslash E_{n,j}$ and about z) and define $f_{n,j}(z)$ as the following Cauchy Integral:

$$f_{n,j}(z) = -\frac{1}{2\pi i}\int_{\Gamma_{n,j}} \frac{f(w)}{w-z}\,dw$$

We need the following lemmas to show that $\gamma(E) = 0$. Proofs are omitted as they are generally simple estimates and/or manipulations of the Cauchy Integral

Garnett's Counter-Example

Necessary Lemmas

Lemma

(a) $\sum_{j=1}^{4^n} f_{n,j} = f$
(b) There is a constant M *such that* $f_{n,j} : \mathbb{C} \setminus E_{n,j} \to \mathbb{C}$ *is holomorphic,* $||f_{n,j}||_\infty \leq M$ *and* $f_{n,j}(\infty) = 0$
(c) $|f'_{n,j}(\infty)| \leq 4^{-n} M \gamma(E)$
Note that the smallest such constant is $M = 1 + \frac{6}{\pi}$.

Lemma

For any $\varepsilon > 0$ *and* $M > 0$*, there exists* $\delta > 0$ *such that for any* f *with* $f : \mathbb{C} \setminus K \to \mathbb{C}$ *(holomorphic),* $||f||_\infty \leq M$, $f(\infty) = 0$ *and* $|f'(\infty)| \geq \varepsilon$ *we have:*

$$\sup_{n,j} 4^n |f'_{n,j}(\infty)| \geq (1+\delta)|f'(\infty)|$$

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Proof that $\gamma(E) = 0$

Proof.

Let E be the four-corners Cantor set and let f be such that $f : \mathbb{C} \backslash E \to \mathbb{C}$ (holomorphic), $||f||_\infty \leq M$, $f(\infty) = 0$ and $a = |f'(\infty)| > 0$ so that by assumption $\gamma(E) > 0$. Choose n_1 and j_1 such that by applying Lemma 5 (Using $\varepsilon = a$ and M as in Lemma 4) we have:

$$|f'_{n_1, j_1}(\infty)| \geq a(1+\delta)4^{-n_1}$$

Since E_{n_1, j_1} is geometrically similar to E we can apply Lemma 5 to f_{n_1, j_1} and choose some (n_2, j_2) such that $|f'_{n_2, j_2}(\infty)| \geq 4^{n_1}|f'_{n_1, j_1}(\infty)|(1+\delta)4^{-n_2} \geq a(1+\delta)^2 4^{-n_2}$. Continuing in this manner, we obtain a sequence (n_k, j_k) with $|f'_{n_k, j_k}(\infty)| \geq a(1+\delta)^k 4^{-n_k}$; however, this contradicts Lemma 4c, yielding the reverse inequality. Therefore $f'_{n_k, j_k} \to 0$ and subsequently $f' \to 0$ so $\gamma(E) = 0$.

□

Outline

Denjoy's Conjecture

History and Introduction

In 1909 Arnaud Denjoy made the following conjecture (and provided an incorrect proof within a year)

Conjecture

(Denjoy) Let $E \subset \mathbb{C}$ be a subset of a rectifiable curve Γ. Then $\gamma(E) = 0$ if and only if $H^1(E) = 0$

- Surprisingly, this conjective is rather difficult to prove; Chronologically we have the following:
  - In 1950, L. Ahlfors and A. Buerling showed that the Denjoy conjecture holds if Γ is a straight line
  - In 1962, L. D. Ivanov showed that the Denjoy conjecture is valid if $\Gamma \in C^{1+\varepsilon}$
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  - Finally in 1977, A. P. Calderón showed that the Cauchy Integral operator[2](for C^1 curves) is bounded on L^p $(1 < p < \infty)$ and it turns out that Denjoy's conjecture is a (long) corollary of this theorem

[2] $\mathscr{L}_{op} = \int_\gamma d\mu(\xi)\, \frac{1}{z-\xi}$

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Denjoy's Conjecture
Preliminaries

- Unfortunately there are quite a few tools from functional analysis needed to prove this theorem; however, I will try to summarize the main ideas in the proof that center around the L^p-boundedness of the Cauchy transform, $\mathscr{C}_\mu = \int \frac{d\mu(\xi)}{\xi - z}$ for $z \notin E$ and spt$\mu \subset E$.

Definition

Suppose we are given a linear operator T. Now define a linear operator T_ε in the principal value sense (like the Hilbert Transform) so that $T_\varepsilon\{f(x)\} = \int_{|x-y|>\varepsilon} f(y)K(x,y)d\mu(y)$ and $\lim_{\varepsilon \downarrow 0} T_\varepsilon = T$. Now define the T^* operator by $T^* f(x) = \sup_{\varepsilon > 0} |T_\varepsilon f(x)|$. In essence the T^* operator acts as an upper bound for T.

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Preliminaries

Definition

A holomorphic function f on a set $\Omega \subset \mathbb{C}$ is said to belong to the class $E^p(\Omega)$ if there exists a sequence of rectifiable Jordan cruves, $\Gamma_1, \ldots, \Gamma_n, \ldots$ in Ω such that $\Gamma_n \to \partial\Omega$ (so that eventually Γ_n surrounds every compact subdomain of Ω) and $\int_{\Gamma_n} |f(z)|^p |dz| \leq C < \infty$

Definition

A d-dimensional *Lipschitz Graph* is a subset of $\mathbb{R}^n$ of the form $\{(x, f(x)) : x \in \mathbb{R}^d\}$ where $f : \mathbb{R}^d \to \mathbb{R}^{n-d}$ is a Lipschitz map or is the image of such a subset by rotation.

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Preliminaries

- N ote that $E^p(\Omega)$ is essentially an analogue of the Hardy Spaces $H^p(\mathbb{D})$ (i.e. the set of functions such that $\sup_{0<r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}} < \infty$) to an arbitrary set Ω
- If $f \in E^2(\Omega)$ then $f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z} dH^1(\xi)$ for $z \in \Omega$. This implies that the Cauchy Transform on $\Gamma = \partial\Omega$ exists for all $f \in E^1(\Omega)$. In order to show the boundedness of this transform on Lipschitz Graphs, we need to use a variant of the argument used to show that $L^p \to L^p$; however, since E^1 is analagous to H^1 the duality argument needs to be adjusted as the $(E^1)^* = BMO$.
- Now note that P. Garabedian showed that $\gamma(\Gamma)^{1/2} = \sup\{|h'(\infty)| : h \in E^1(\Omega), ||h||_{E^2(\Omega)} \leq 1\}$ by solving the dual extremal problem $\inf\{||g||_{E^1(\Omega)} : g \in E^1(\Omega), g(\infty) = 1\}$. The proof involve the fact that the Szëgo kernel $K_x(y) = \frac{1}{1 - \overline{xy}}$ is the reproducing kernel of $E^2(\Omega)$ (i.e. the kernel such that $< f, K_x(y) >= f(x + iy)$).

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Denjoy's Conjecture
Outline

- We will show the contrapositive, i.e. that $H^1(E) > 0 \Rightarrow \gamma(E) > 0$.
- As the Cauchy Transform is bounded, we simply need to find an example of a function whose Cauchy Transform relies on $H^1(E) \Rightarrow \chi_E$ so that $\gamma(E) > 0$

Denjoy's Conjecture

Super Sketch(y) Proof

Proof.

Let Γ be Lipschitz graph and let $f \in E^2(\Gamma, ds)$ (where the measure ds refers to arc length). Define the following integral transform, $\mathscr{C}$:

$$\mathscr{C}f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{z-\xi}\,dH^1(\xi)$$

where H^1 is the 1-dimensional Hausdorff Measure (not $H^1(\Omega)$ the Hardy Space). We know thatt the operator $\mathscr{C}$ is bounded[a] on $E^2(\Gamma, ds)$. Therefore, $\mathscr{C}f$ has boundary values $\mathscr{C}^*f$ on Γ and $\mathscr{C}^*f \in L^2(\Gamma, ds)$ so that $\mathscr{C}f \in E^2(\Omega)$ where $\Omega \subset \mathbb{C}$ with boundary Γ. Now let $E \subset \Gamma$ be compact and approximate $\tilde{E}$ by a finite (cover) of subarcs of Γ. Now note that $\mathscr{C}\chi_{\tilde{E}} \in E^2(\Omega)$ as $\chi_{\tilde{E}} \in L^2(\Gamma, ds)$ and subsequently

$$|\mathscr{C}'_{\chi_{\tilde{E}}}(\infty)| = \frac{1}{2\pi}H^1(\tilde{E})$$

Therefore from the Garabedian formula, $|\mathscr{C}'_{\chi_{\tilde{E}}}(\infty)| = \frac{1}{2\pi}H^1(\tilde{E}) \leq |h'(\infty)|^2 = \gamma(\Gamma)$ so that if $H^1(\tilde{E}) > 0$ then $\gamma(\tilde{E}) > 0$. The monotonicity of γ implies that $\gamma(E) > 0$.

$\square$

[a] Steve's Presentation on 04/28/09

Vitushkin's Conjecture

- An even more far-reaching generalization of Denjoy's Conjecture is Vitushkin's Conjecture (1968) which is:

Conjecture

Let $E \subset \mathbb{C}$ be a compact set with $H^1(E) < +\infty$. Then E is removable for bounded analytic functions if and only if E is purely 1-*unrectifiable*

- G. David proved this theorem in 1998 using his *Tb* theorem; unfortunately this theorem's proof is a one-hundred page paper full of Harmonic analysis, which makes the proof impossible to present.

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Summary

- For the most part, Painlevé's Problem has been solved using the useful notion of Analytic Capacity
- Crucial results in the analytic characterization of removable sets have been found
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For Further Reading I

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