Senior Thesis Problem Statement

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Abstract

We briefly summarize the mathematical and physical motivation for the Anti-de-Sitter Space/Conformal Field Theory (AdS/CFT) Correspondence and its relationship to the construct of an infinite family of 5-dimensional Sasaki-Einstein Manifolds. A statement of the (thesis) problem at hand is then described.

1 Introduction

The Anti-de-Sitter Space/Conformal Field Theory correspondance is a derived result of String Theory that posits a geometric relationship between Gauge Theories and the space-times of General Relativity [2, 22]. Physically, the AdS/CFT correspondance states that if one has the n-dimensional Anti de Sitter Space, $M = AdS_n$ as a model of spacetime, then the partition function associated to a given conformal structure on M is determined by the vacuum expectation value of a linear functional J on $\mathrm{Sym}(2, \partial M)$, the set of symmetric 2-tensors on ∂M. On the other hand, the correspondence conjectures a mathematical relationship between an asymptotically hyperbolic, Einstein metric g on a $n+1$-dimensional manifold with boundary M and the restriction to ∂M of the n^{th} term in the Fefferman-Graham expansion of geodesically equivalent Einstein metric. In physics, such a relationship arose from the discovery that certain 5-dimensional Riemannian Manifolds X_5 give rise to a string background $AdS_5 \times X_5$ such that the choice of metric g on X_5 could determine the central charge of a Conformal Field Theory on $\partial(AdS_5)$.

The goal of this paper is to determine the spectrum of the Hodge-de Rham Laplacian on forms for a particular family of 5-manifolds X_5 that satisfy both physical and mathematical constraints unique to this physical "duality" problem. In particular, there is a new infinite family of Sasaki-Einstein 5-manifolds $Y^{p,q}$ that appear to be the best candidates for non-trivial application of the AdS/CFT conjecture.

This paper is structured so that both the physical and mathematical problems at hand are explicitly described. §2 will provide a brief introduction to the breadth of mathematics required to elucidate the AdS/CFT correspondance while providing the analyical definition of what exactly the correspondance is claiming. §3 will provide a review of the physics involved and the motivations for studying the manifolds $Y^{p,q}$. §4 will provide a brief description of Sasaki-Einstein manifolds as well as the explicit metrics involved. §5 will conclude this paper with a description of the problem at hand.

2 Formal statement of the AdS/CFT correspondance

2.1 Formulating a Conformal Field Theory

While many mathematicians tend to look at String Theory as most similar to Algebraic Geometry, the AdS/CFT correspondance provides both an analytic and differential geometric point-of-view of String Theory and Conformal Field Theories. Before arriving at the more analytic description of the correspondance, let's consider the bare necessities for a mathematical formulation of a Conformal Field Theory. Let M be an n-dimensional Manifold that will serve as our *Configuration Space* of possible physical states. In order to describe a *Conformal Field Theory*, one needs a linear functional[1] $J : \mathcal{T}M \otimes \mathcal{T}^*M \to R$, called an *action*, that is (locally) invariant under the group of conformal transformations of $\mathbb{R}^n, \mathrm{Conf}(\mathbb{R}^n)$. In order to determine quantities of interest, one would ideally like to compute how the action (which contains all the physics) describes the time-evolution of a particle over a specific path $\gamma : [0,1] \to M$. However, this implies that we are integrating over the set of all C^∞ paths γ. This set is necessarily and infinite-dimensional space, since for each chart $\varphi_\alpha : U_\alpha \subset M \to \mathbb{R}^n$, the set of all paths contained in U_α is $C^\infty([0,1], \mathbb{R}^n)$ which is an infinite-dimensional Banach Space. It is a standard result of Real Analysis that there exists no translational-invariant, infinite-dimesional Banach Space and as such a "sum over all paths" is not mathematically well-defined. A crucial symmetry in physics in Poincaré Symmetry, which is invariance of J under the action of the n-dimensional Poincaré Group $\mathsf{SO}(n) \rtimes \mathbb{R}^n$. Intuitively, this invariance says that the energetics of a physical system, as prediced by the action J, should not change under a change of origin (translation) or a rotation. As such it is physically necessary that the action J be translational-invariant, so a precise formuation of the "expectation of physical event A with respect to the constraints contained

[1] Notation: $\mathcal{T}M$ is the tensor algebra of TM amd $\mathcal{T}^*M$ is the tensor algebra of T^*M

in J" is currently unknown. However, physicsists have come up with clever ways to approximate such "non-existent" expectations.

In the physics literature, the expectation value of the functional J is typically presented in the form of a Feynman Path Integral that sums over the restriction to ∂M of "all Einstein metrics" on M. While this initially seems to be an ill-defined object, G. Segal and M. Kontsevich have proposed a mathematical definition[2] of the expectation with respect to a Gaussian measure on Teich(M), the Teichmüller Space of M [26]. From an analytic perspective this should be well-defined, since the Teich(M) is in general a Banach Space. This paper will not be concerned with the ambiguity in the expectation value since the main goal of this thesis is analytic in nature.

2.2 Analytic Definition of the Correspondance

We will assume that any manifold M defined is of class C^∞. Moreover, we will restrict ourselves to the category of *Einstein Manifolds* (M, g_E), which have $\text{Ric}(u,v) = \kappa g_E, u, v \in TM, \kappa \in \mathbb{R}$. Let us first define what exactly a Conformally Compact metric is [7]:

Definition 2.1. *Suppose that we have an $n+1$ dimensional Riemannian Manifold with non-empty boundary, $(\overline{M}, \overline{g})$. Let M be the interior of $\overline{M}$ and let ∂M be the boundary of $\overline{M}$. A complete Riemannian metric g on M is* ***conformally compact*** *if there exists a function $\rho \in \Omega^0(\overline{M})$ such that $\overline{g} = \rho^2 g$, $\rho^{-1}(0) = \partial M$ and $d\rho \neq 0$ on ∂M. Such a function ρ is called a* ***defining function***[3] *for the pair $(M, \overline{M})$*

Define $\gamma := \overline{g}|_{\partial M}$ which is the boundary metric on $\overline{M}$ and let $\mathcal{M}, \mathcal{M}_\partial$ be the moduli spaces of metrics and boundary metrics on conformally compact, $n+1$ dimensional Einstein manifolds with non-empty boundary, respectively. The choice of γ is in general not unique because there can be many defining metrics for a given $\overline{M}$. Instead, we will work with boundary metrics γ defined up to conformal transformation; that is, we consider the equivalence class $[\gamma]$ under the relation,

$$\gamma', \gamma \in \mathcal{M}_\partial, \gamma' \sim \gamma \iff \exists f : \overline{M} \to R, \text{Im}(f) \subset (0, \infty), \gamma' = f\gamma$$

Note that since we are considering boundary metrics up to conformal transformation we are indeed only interested in $\mathcal{M}_\partial := \text{Teich}(M)/\text{MCG}(M)$ as opposed to $\text{Teich}(M)$, since diffeomorphic manifolds will have the same conformal structure. Moreover, note that the physical interpretation of the boundary metric $[\gamma]$ is embodied in Penrose's notion of *Conformal Infinity*. The idea is that the causal structure of spacetime (i.e. whether a geodesic is timelike, null or spacelike) is preserved under conformal transformations so that singularities (such as Black Holes) can be more easily analyzed.

In order to simply the mathematical content of the AdS/CFT correspondence, assume that we are in a regime that satisfies the vacuum Einstein equations, $G_{\mu\nu} = 0$, where $G_{\mu\nu}$ is the Einstein Tensor. We define the *Einstein-Hilbert action*, $S_{EH} : \mathcal{M} \to \mathbb{R}$ by

$$S_{EH}(g) := \int_M K dV_g$$

where K is a Gaussian Curvature of (M, g) and dV_g is the volume form associated to g. In coordinates, this reduces to the more recognizable action, $S_{EH} = \int_M d^{n+1}x\sqrt{g}R$, where R is the Ricci Scalar Curvature. Heuristically, one can say that the AdS/CFT correspondence takes boundary data $(\partial M, [\gamma])$ and derives a partition function for $(\overline{M}, \overline{g})$. Finally let $\mathcal{M}_{(\partial M, [\gamma])} := \left\{g \in \mathcal{M} | \exists \rho \in \Omega^0(\overline{M}) \ni \overline{g} = \rho^2 g, \gamma = \overline{g}|_{\partial M}\right\}$ so that given these definitions, the AdS/CFT correspondence can be summarized by the following equation:

$$Z_{CFT}(\partial M, [\gamma]) = \sum_{g \in \mathcal{M}_{(\partial M, [\gamma])}} e^{-S_{EH}(g)} \tag{1}$$

where Z_{CFT} is the partition function associated to a conformal field theory associated with the conformal structure $[\gamma]$ on ∂M. Note that the principle concept behind the sum on the right hand side is that we are summing over *all* Conformally Compact Einstein Manifolds (M, g) *given* the boundary data $(\partial M, [\gamma])$. However, it is clear that (1) doesn't provide any intuition as to the analytic aspects of the AdS/CFT conjecture. This is where the Fefferman-Graham expansion and the asymptotic hyperbolicity of the metric come into play. Recall that a metric g is hyperbolic if it has constant, negative sectional curvature. We can now define what an asymptotically hyperbolic metric is:

[2]Segal and Kontsevich formulate Conformal Field Theory in terms of Cobordism Classes of compact, connected 2-manifolds that resemble the worldsheets of String Theory. This functorial definition constructs Conformal Field Theory in terms of a Generalized Cohomology Theory, with two compact, connected Riemann Surfaces M, N deemed equivalent iff M is cobordant to N. I have a bit of trouble connecting this description with the analytic aspects of Field Theories, so I've chosen to ignore it for the most part.

[3]Note that we assuming that ρ is of class C^∞. This definition has been expanded in the following way: A metric g is said to be $L^{k,p}$ or $C^{m,\alpha}$ conformally compact if there exists a defining function such that $\overline{g}$ has an $L^{k,p}$ or $C^{m,\alpha}$ extension to $\overline{M}$, where $L^{k,p}$ is the $(k,p)-$Sobolev Space and $C^{m,\alpha}$ is the $(m,\alpha)-$Hölder space. See [3] for details

Definition 2.2. *Again suppose that we have an $n+1$ dimensional Riemannian Manifold with non-empty boundary, $(\overline{M}, \overline{g})$. Let M be the interior of $\overline{M}$ and let ∂M be the boundary of $\overline{M}$. We say that g is* ***asymptotically hyperbolic*** *if $\overline{g}|_{\partial M}$ is hyperbolic.*

Now there are some interesting properties about asymptotically hyperbolic Einstein metrics, namely that they have a canonical form in which they can be expressed as a cone over a a family of hypersurface metrics. This splitting comes via the Gauss Lemma and represents the first analytic formulation of the AdS/CFT conjecture. Let detail this process a bit. Suppose that we choose a defining function $\rho(x) = d_{\bar{g}}(x, \partial M)$ in a collar neighborhood[4] of ∂M. A defining function of this type is called a *geodesic* defining function, since the minimization of $\rho(x)$ will solve a geodesic-like problem on the conformal infinity $(\bar{M}, \bar{g})$.

It is possible to prove that given a boundary metric $[\gamma]$ in the conformal infinity of (M, g) there exists a unique geodesic defining function ρ_γ that has γ as the boundary metric on $(\bar{M}, \bar{g})$ [4]. The proof boils down to reformulating the uniqueness problem in terms of a Cauchy problem on the collar neighborhood U. Recall the Gauss Lemma of Riemannian Geometry [29]:

Lemma. *(Gauss) The radial geodesic through the point $p = \exp(\xi), \xi \in T_pM$ is orthogonal to the Riemann Hypersurface Σ_p that passes through p.*

If we choose $p \in \partial M$, then this says that in some normal (or inertial) neighborhood U of p, we can write the metric $\bar{g}|_U = dt^2 + g_{\Sigma_p}|_U$. Since we can choose a boundary metric up to conformal transformation, the choice of a geodesic defining function gives a conformal class of metrics such that *globally* we have $\bar{g} = dt^2 + g_t$, where g_t is a family of metrics on hypersurfaces where $t = t'$. Now we can define the *Fefferman-Graham expansion* of the metric $\bar{g}$ as the truncated Taylor Series expansion of g_t. Explicitly for an n-dimensional Riemannian Manifold (M, g) with conformal infinity, this expansion is [14]:

$$g_t = g_0 + tg_1 + t^2 g_2 + t^3 g_3 + \ldots + t^n g_{(n)} + O(t^{n+\alpha}) \tag{2}$$

Given the above background, the conjecture is effectively the dimension $n = 5$ version of the following dimension $n = 4$ theorem [3, 6]:

Theorem. *If $\dim M = 4$ and the boundary metric γ is of class[5] $C^{7,\alpha}$. Then the pair $(\gamma, g_{(3)})$ on ∂M uniquely determined an Asymptotically Hyperbolic Einstein metric up to local isometry. This means that if g^1, g^2 are two AH Einstein metrics on manifolds M_1, M_2 with $\partial M = \partial M_1 = \partial M_2$ such that with respect to the aforementioned compactifications $(\bar{M}_1, \bar{g}_1), (\bar{M}_2, \bar{g}_2)$, we have:*

$$\gamma^1 = \gamma^2 \quad \textit{and} \quad g^1_{(3)} = g^2_{(3)}$$

then g^1, g^2 are locally isometric and M_1, M_2 have diffeomorphic universal covers.

This effectively says that given a boundary metric γ and an n-th order approximation of the interior metric g, we can compute g up to local isometry. For physical purposes, one desires knowledge of the null and timelike geodesics, so that this uniqueness up to local isometry is "good enough." Moreover, the hoice of boundary metric γ places essentially defines the boundary Energy-Momentum Thensor, so the above theorem will yield the Energy-momentum tensor for the entire spacetime M. The case of $\dim M = 5$ appears to not have been proved completely yet. In the next section, we will show how a choice of String Theory background (a geometric constaint) fixes the equivalence class $[\gamma]$ so that the above theorem can be used.

2.3 Geometric Description of the Correspondance

As mentioned in the introduction, the AdS/CFT correspondance is closely tied together with Complex Geometry and String Theory. Let's first give a short description of the geometric structures associated with String Theory. String Theory purports that if strings that obey known symmetries[6] as well as supersymmetry exist, then the total spacetime manifold M must be 10-dimensional. As such, most early formulations of string theory assumed that $M = \mathbb{R}^{1,3} \times X_6$, where X_6 is either a 6-dimensional real manifold or a 3-dimensional complex manifold. The idea is that if $X_6 \hookrightarrow \mathbb{R}^{12} \cong \mathbb{C}^6$ is contained in a ball of radius r in $\mathbb{R}^{12}$ or $\mathbb{C}^6$, then as $r \to 0$, M would begin to look like Minkowski Space, $\mathbb{R}^{1,3}$. This

[4] Recall that a collar neighborhood U of an n-manifold M with boundary is an open set $U \subset \mathbb{R}^{2n}$ such that U is diffeomorphic to $\partial M \times [0, \epsilon)$. The Whitney Embedding Theorem guarantees the existence of such a neighborhood in $\mathbb{R}^{2n}$.

[5] Recall that the Hölder Spaces $C^{k,\alpha}(\Omega), \Omega \in \mathbb{R}^n$ are the topological vector spaces of functions $f : \Omega \to \mathbb{R}$ that are k times continuously differentiable and such that f is Hölder continuous with exponent α. This means that $\forall x, y \in \Omega, |f(x) - f(y)| \leq C|x - y|^\alpha$.

[6] Recall that for a symplectic manifold (M, ω), the Lagrangian formulation of physics on M allows for all physical quantities to be written in terms of a linear functional that can be extremized via an Euler-Lagrange equation. For purposes of this paper, we will consider the Lagrangian to be a linear functional $\Lambda : L^2(\bigwedge^\bullet M, dm) \to \mathbb{R}$. In this situation, a *symmetry* of a Lie Group G with an action on $\Gamma(TM)$ such that $\forall f \in L^2(\bigwedge^\bullet M, dm)$, f, ω are invariant under the flows of $g \cdot \vec{v}, \vec{v}$ where $\vec{v} \in \Gamma(TM)$

means that all known physics, which requires a background of Minkowski space or a locally-Minkowski space (i.e. a Lorentzian Manifold), could be preserved if the embedding radius r of X_6 is quite small. The idea then was that the "worldsheet" of the string would be an embedded submanifold of X_6, *independent* of $\mathbb{R}^{1,3}$. The strings would effect the evolution of a physical system in M, for example, since an operator such as the Laplacian would have contributions from X_6, but it would be disjoint from it. Physicists denote conserved quantities of the evolution of a physical system in X_6 (which is guided by extremizing a linear functional $J : L^2(\bigwedge^\bullet X_6 \to \mathbb{R})$ as *Stringy Corrections.* Note that because Complex Manifolds inherently have an integrable structure with an easy way to check symplectic and differentiable structure integrability (i.e. via the Newlander-Nirenberg Theorem), almost all models for X_6 were complex.

However in the early-1990s, there was some compelling evidence that suggested that one should consider spacetime manifolds M that decompose as $M = X \times Y$ with $\dim X = \dim Y = 5$. The idea stemmed from the fact that the 5-dimensional Anti de Sitter space, AdS_5 can be viewed as the Lorentzian analogue of the hyperbolic space $\mathbb{H}^n$ [2]. More precisely, we can define AdS_5 as via the locus of the quadratic polynomial f on $\mathbb{R}^6$ with coordinates $(x_1, x_2, x_3, x_4, S, T)$,

$$f(x_1, x_2, x_3, x_4, S, T) := x_1^2 + x_2^2 + x_3^2 + x_4^2 - S^2 - T^2 + 1 \tag{3}$$

It is clear that f is a submersion, since $dF(\vec{x}) = (2x_1, 2x_2, 2x_3, 2x_4, -2S, -2T)$ so that the submersion level set theorem says that this zero locus is an embedded, 5-dimensional submanifold of $\mathbb{R}^6$. Note that the induced metric on AdS_5 is $g_{AdS_5}(\partial_1, \ldots, \partial_4, \partial_S, \partial_T) = dx^1 + \ldots dx^4 - dS^2 - dT^2$, so that AdS_5 is the Lorentzian analogue of the Hyperbolic Space $\mathbb{H}^5$. Moreover, note that the surfaces with constant x_1, T are copies of $\mathbb{R}^{1,3}$ As such we can view AdS_5 as a set of hypersurfaces $\Sigma_{x_1,T}$ that are isometric to $\mathbb{R}^{1,3}$ with a scaling factor of $R^2 := x_1^2 - T^2$. The idea is that we are using one of the six remaining (and remember, required) dimensions to construct a space that heuristically "warps" Minkowski Space. The heuristic idea is that if $X = AdS_5$ and Y is a 5-dimensional manifold, then it is possible for a stringy correction to depend on which $\Sigma_{x_1,T}$ is chosen. This means that a string theory effect can somehow affect the spacetime manifold that we perceive, in this case $\mathbb{R}^{1,3}$.

As it turns out, our current knowledge of physics and our desire to have supersymmetry heavily constrains the choice of manifold Y we choose. Let us briefly review some of the main assumptions in the decomposition of spacetime as $M = \mathbb{R}^{1,3} \times X_6$. The main condition enforced is that X_6 has a Kähler metric and is Ricci-Flat[7]. The Kähler metric can be heuristically justified on the grounds that classical mechanics formally requires a symplectic form while quantum mechanics requires symmetries to be preserved under unitary transformations. The Ricci-Flat condition implies that our X_6 satisfies the vacuum Einstein equations $G(\hat{e}_\mu, \hat{e}_\nu) = T(\hat{e}_\mu, \hat{e}_\nu)$ where G is the Einstein 2-tensor, $G(\hat{e}_\mu, \hat{e}_\nu) := \mathrm{Ric}(\hat{e}_\mu, \hat{e}_\nu) - \frac{1}{2} g_{X_6}(\hat{e}_\mu, \hat{e}_\nu) R$ and $T \in TM \otimes TM$ is a symmetric 2-tensor[8]. A complex manifold that is Kähler and Ricci-Flat is known as a *Calabi-Yau Manifold.* Physicists have been interested in these manifolds precisely because they are compatible with the symmetries of nature and serve as a good starting point. However, while Yau proved that such manifolds exist, no explicit metric has ever been found so this has made analysis on these spaces difficult. For completeness and for use in §4, let us give some equivalent definitions of a Calabi-Yau Manifold:

Theorem. *For a compact complex n-manifold (M, g), the following are equivalent:*

- *M is a Calabi-Yau*
- *$Hol_g(M) \subset \mathsf{SU}(n)$*
- *The first Chern Class of M, $c_1(M)$ vanishes*
- *The canonical bundle of M is trivial*
- *M admits a global, non-vanishing holomorphic n-form*

The AdS/CFT correspondance effectively conjectures that for a certain class of 5-manifolds Y, we can still preserve the symmetries required to have a well-defined string theory and that the choice of metric on Y *uniquely* determines the conformal boundary $[\gamma]$ of AdS_5 [2, 4, 22]. The conjecture was initially formulated with $AdS_5 \times S^5$, where S^5 is given the round metric and with $AdS_5 \times S^2 \times S^3$, where $S^2 \times S^3$ is given the homogeneous metric.[9] However, these examples are considered trivial in that they admit free, proper $U(1)$ actions so that given a $U(1)$ action, any quotient of S^5 or $S^2 \times S^3$ by this action will be a Kähler-Einstein Manifold with positive curvature [23]. However, physically this is too restrictive as Maxwell's Laws are usually mathematically generalized to the set $U(1)$-valued Ehresmann connections. This means that one only requires a periodic $U(1)$ orbit as opposed to a free $U(1)$ orbit. Recently, a new infinite class of

[7] This means that the Ricci Scalar $R := \mathrm{tr}(\mathrm{Ric}(\hat{e}_i, \hat{e}_j))$, for any local frame $\{\hat{e}_i\}$ vanishes
[8] Physically, this is the *Energy-Momentum Tensor*
[9] Recall that $S^2 \approx \mathsf{SO}(3)/\mathsf{SO}(2)$ and $S^3 \approx \mathsf{SU}(2)$ so that the product can be considered a homogeneous space

5-manifolds that are compatible with AdS/CFT correspondance and admit both free and non-free $U(1)$ orbits have been constructed. This class of manifolds subsumes $S^5, S^2 \times S^3$ and represent *Sasaki-Einstein Manifolds*. These manifolds are defined and analyzed in §4.

3 Why Sasaki-Einstein Manifolds?

Long Story Short: Physical contraints, such as the desire to preserve supersymmetry and Ricci-flatness of the metric on X_5, led String Theorists to consider Sasaki-Einstein Manifolds as a good model of the AdS/CFT Correspondance. From Yau's Theorem, one knows that if a complex n-manifold M has $\text{Hol}(M) \subset SU(n)$, then M admits a Ricci-flat, Kähler metric. In order to find a 5-dimensional counterpart to AdS_5, the authors of [1] showed that the most natural 5-manifold to consider is the Sasaki-Einstein Manifold. Moreover, it can be shown that Sasaki-Einstein Manifolds admit a Killing Spinor (for a "canonical" spin bundle that one can define on a Sasaki-Einstein Manifold) which implies that it is possible to preserve supersymmetry.
To be completed!

4 The geometry of the the $Y^{p,q}$ Manifolds

4.1 Brief Overview of Sasaki-Einstein Manifolds

In this section, we will only consider manifolds of dimension 5 and higher. Let us start with the most cogent definition of a Sasaki-Einstein Manifold:

Definition 4.1. *An odd-dimensional, compact, real Riemannian manifold (M, g) is* ***Sasaki-Einstein*** *iff it is Einstein and its metric cone $(C(M), \overline{g}), C(M) \cong \mathbb{R}_+ \times M, \overline{g} = dr^2 + r^2 g_M$ is Kähler and Ricci-flat, or in other words Calabi-Yau. We will naturally identity M via as $\{1\} \times M \subset C(M)$*

This brief introduction will will follow §1 of [28] and portions of Chapters 3,6 and 11 of [8], which is relatively recent and complete monograph on Sasakian Geometry. Since a Sasaki-Einstein Manifold has a Kähler cone, it inherits many of the nice features of Kähler and Symplectic manifolds. In particular, the odd dimensional cousins of Kähler and Symplectic geometries are *CR* and *Contact Geometries*, respectively. One can think of contact geometry as an odd-dimensional analogue of symplectic geometry, inspired by classical mechanics with a configuration space that also depends on time. More formally, we define a *Contact Structure* (M, η) on an $2n + 1$ manifold M with one-form η if $\eta \wedge (d\eta)^n$ is a volume form. In that case of a Sasaki-Einstein Manifold M, if $\omega_{\overline{g}}$ is the Kähler form of $C(M)$, then one can prove ([8], §6.4-6.5) that the Kähler potential can always be put in the form $\omega_{\overline{g}} = d(r^2\eta)$ for a one-form η on $C(M)$. This means that $r^2\eta$ is a Kähler potential and moreover that for $r = 1$, we get a non-vanishing one-form on M. Since $d\eta$ is also globally non-vanishing, by *Khlerity*, we get an induced contact structure on M. The following proposition is quite important (from [8], §6.1):

Proposition. *On a contact manifold (M, η) of dimension $2n + 1$, there exists a unique vector field ξ called the* ***Reeb Vector Field*** *satisfying the conditions, $\xi \lrcorner \eta = 1, \xi \lrcorner d\eta = 0$*

The proof is relatively straightforward and gives some intuition about where ξ comes from, so let's go through it:

Proof. Since (M, η) is a contact manifold[10] we have a volume form $\eta \wedge (d\eta)^n$. By the musical isomorphism $TM \cong T^*M$, there exists a unique $\xi \in \Gamma(TM)$ such that $\xi \lrcorner \eta = 1$ so that $\xi \lrcorner \eta \wedge (d\eta)^n = (d\eta)^n$. since $(d\eta)^n = \underbrace{d\eta \wedge d\eta \cdots \wedge d\eta}_{n}$, is alternating this means that $\xi \lrcorner (d\eta)^n = 0$. ∎

For a Sasaki-Einstein manifold, the Reeb Vector Field takes a rather simple form, via a slightly opaque construction. Consider the *homothetic vector field*, ζ on $C(M)$ is simply $\zeta := r\partial_r$. Let $\nabla, \overline{\nabla}$ be the Levi-Civita connections associated to $g, \overline{g}$, respectively. In order to construct a vector field on M that extends naturally to a vector field on $C(M)$, we need to ensure that we have a real analytic vector field on M. Let's first look at the transport behavior of ζ via the following formulas [28] hold for $X, Y \in \Gamma(\{1\} \times M) \hookrightarrow \Gamma(C(M))$:

$$\overline{\nabla}_\zeta \zeta = \zeta \tag{4}$$
$$\overline{\nabla}_\zeta X = \overline{\nabla}_X \zeta = X \tag{5}$$
$$\overline{\nabla}_X Y = \nabla_X Y - g(X, Y)\zeta \tag{6}$$

[10] When the phrase 'contact manifold' is used, we will mean an *strict contact manifold* in the sense of [8], page 181

Heuristically equations (4),(5) imply that a vector field X on the base M doesn't change as we move it up the cone via ζ. Now since $C(M)$ is Kähler, we know that the Almost Complex Structure $J : TC(M) \to TC(M)$ associated to $C(M)$ is parallel, i.e. $\overline{\nabla} J = 0$. Using these facts we can prove the following claim:

Claim 1. *The homothetic vector field ζ is real analytic, i.e. $\mathcal{L}_\zeta J = 0$.*

Proof. Since $\overline{\nabla}, \nabla$ are Levi-Civita connections, the torsion tensor for both connections vanishes. This means that $\mathcal{L}_\zeta J = \overline{\nabla}_\zeta J - \overline{\nabla}_J \zeta$. Using (5), it is clear that this vanishes. Similarly if we restrict to the hypersurface $\{1\} \times M$, then using (5), (6), we have,

$$\mathcal{L}_{\zeta|_M} J = \nabla_{\zeta|_M} J - \nabla_{J|_M} \zeta = \overline{\nabla}_{\zeta|_M} J + g(\zeta|_M, J)\zeta - \overline{\nabla}_J \zeta|_M - g(J, \zeta|_M)\zeta = 0$$

where the last equality holds since the metric is symmetric. □

Now define $\xi = J(r\partial_r)$. We now will show that the $\eta = \frac{1}{r^2}\xi^\flat$ is a contact form. Recall that given an almost Hermitian Manifold (M, g, J), we define the Kähler form $\omega_g : TM \otimes TM \to \mathbb{R}$ by $\omega_g(X, Y) := g(X, JY)$. The musical isomorphism gives the formula $\eta(X) = \frac{1}{r^2} g(X, \xi) = \frac{1}{r^2}\omega_g(X, r\partial_r)$. Since $r^2\eta$ is the Kähler potential, it is clear that η is non-vanishing and more over, since the Kähler form is also symplectic, $(d\eta)^n$ is also non-vanishing. As such, $\eta \wedge (d\eta)^n$ is a non-vanishing $2n+1$-form, or in other words it is a volume form.

To Be Finished:
What's left to do:

- Show that $\frac{1}{2}r^2$ is the Kähler potential for the cone
- Define the metric contact structure
- $\exists$ a Global Killing Vector that comes from the Global Killing Spinor on the Cone
- Reeb Foliation
- Define Regularity, Reeb Foliations

4.2 The $Y^{p,q}$ metrics

4.2.1 Background

Until 2004, it was widely believed that irregular Sasaki-Einstein Manifolds did not exist as per a conjecture by Tian and Cheeger [9]. However, in 2004 a landmark paper of Sparks, Martelli, Gauntlett and Waldram [13] constructed an infinite sequence of Sasaki-Einstein metrics on $S^2 \times S^3$ which included irregular and quasiregular Sasaki-Einstein Manifolds. These manifolds, denoted $Y^{p,q}$ are indexed by $p, q \in \mathbb{Z}, (p,q) = 1$. Initially, the metrics were described in coordinates and soon after an argument that relied on a straight-forward application of the Gysin Sequence for the natural $U(1)$ fibration, $U(1) \hookrightarrow Y^{p,q} \twoheadrightarrow Y^{p,q}/U(1)$ to show that $Y^{p,q}$ was homotopically $S^2 \times S^3$ so that Smale's Theorem provides the homeomorphism $Y^{p,q} \cong S^2 \times S^3$. Later, an explicit diffeomorphism between $S^2 \times S^3$ and $Y^{p,q}$ was found [12]. Finally, a paper that generalized the construction of $Y^{p,q}$ to a larger family of metrics that could be defined in any odd dimension $2n+1$ on $S^n \times S^{n+1}$ was completed [10, 11]. One of these larger family of metrics is denoted $L^{p,q,r}, p, q, r \in \mathbb{Z}, 0 \leq p \leq q, 0 < r < p+q, (p,q) = (p,r) = (q,r) = 1$ and it's construction is far less complicated than the original construction in [13]. Note that if $p + q = 2r$, then $L^{p,q,r} = Y^{p,q}$. This section will construct the $L^{p,q,r}$ metric due to simplicity and subsequently, we will restrict ourselves to the Laplacian for the $Y^{p,q}$ case.

4.2.2 Construction

We will follow the methodology of [10], which provides a direct route to the $L^{p,q,r}$ metric from a well-known AdS_n solution of the vacuum Einstein equations with negative cosmological constant. The method of construction can be summarized as follows:

1. Start with the five-dimensional Kerr-de Sitter Black Hole Metrics (in local coordinates) found in [18]. These metrics represent solutions to Einstein's Field Equations that admit charged, rotating black holes
2. Consider the "Euclideanization" of these metrics, which amounts to a formal analytic continuous of a real n-manifold to an almost complex n-manifold (i.e. with real dimension $2n$)
3. Implement a supersymmetry constraint known as the *BPS Scaling Limit*

4. Compute the Killing Vectors and Killing Spinors[11] associated to the given metric

The five-dimensional Kerr-de Sitter metric for a rotating, charged black hole (as per Hawking, et. Al, [18]) in local coordinates $(t, \phi, \psi, r, \theta)$ on an open set $U_\alpha \subset AdS_5$ is:

$$g(\partial_t, \partial_\phi, \partial_\psi, \partial_r, \partial_\theta) = -\frac{\Delta}{\rho^2}\left(dt - \frac{a\sin\theta}{\Xi_a}d\phi - \frac{b\cos^2\theta}{\Xi_b}\right)^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(a\,dt - \frac{(r^2+a^2)}{\Xi_a}d\phi\right)^2 + \frac{\Delta_\theta\cos^2\theta}{\rho^2}\left(b\,dt - \frac{r^2+b^2}{\Xi_b}d\psi\right)^2$$
$$+\frac{\rho^2}{\Delta}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{(1+r^2l^{-2})}{r^2\rho^2}\left(ab\,dt - \frac{b(r^2+a^2)\sin^2\theta}{\Xi_a}d\phi - \frac{a(r^2+b^2)\cos^2\theta}{\Xi_b}d\psi\right)^2 \tag{7}$$

where we have:

- a,b are the conserved quantities of the $\mathsf{SO}(4) \cong \mathsf{SU}(2)\times\mathsf{SU}(2)$ symmetry in this metric. Effectively, they scale the Killing Vectors that are the infinitesmal generators of these symmetries
- l is another conserved quantity from the Killing Vector that corresponds to the conserved Energy of the system
- $\Delta = \frac{1}{r^2}(r^2+a^2)(r^2+b^2)(1+r^2l^{-2}) - 2M$
- $\Delta_\theta = (1 - a^2l^{-2}\cos^2\theta - b^2l^{-2}\sin^2\theta$
- $\rho^2 = (r^2 + a^2\cos^2\theta + b^2\sin^2\theta)$
- $\Xi_a = (1 - a^2l^{-2})$
- $\Xi_b = (1 - b^2l^{-2})$

Now we can (somewhat) informally extend this metric (at least locally) to the complexification $U_\alpha \otimes \mathbb{C} \approx \mathbb{R}^n \otimes \mathbb{C}$. To see why this is feasible, consider the metric localized in local coordinates over a coframe $\{dx^i(y)\}_{i=1}^n$ for $y \in U_\alpha$ to be defined as as $g(y) = g_{ij}(y)dx^idx^j$. The idea is that since we are complexifying the chart (U_α, ψ_α), we can also complexify the local trivialization of $\mathrm{Sym}(2, M)$ over U_α. Explicitly, the authors of [10] do this as follows. Consider the following coordinate transformations:

$$\tau := \frac{\sqrt{\lambda}t}{i}, \lambda := -l^2, a' := -ia, b := -ib \tag{8}$$

Now the next simplification step requires taking a limit that relates a, b, r to λ. Effectively, this ties all of the three open parameters a, b, l as well as one coordinate to a single length scale λ. This will immensely simply the metric allowing us to compute the new simplified Killing Vectors. These transformations come from supersymmetry considerations as well as conservation of mass and energy. These transformations represent the *Bogomol'nyiâĂŞPrasadâĂŞSommerfield Limit* (BPS Limit) which is a way to bound the conserved quantity derived from the t or τ coordinates (Energy). Effectively, we are scaling our free parameters as well as our r coordinate in such a way that the energy E goes to the BPS Limit as one takes a limit $\epsilon \downarrow 0$, for a perturbation ϵ. Explicitly, these transformations are:

$$a = \lambda^{-1/2}\left(1 - \frac{1}{2}\alpha\epsilon\right), b = \lambda^{-1/2}\left(1 - \frac{1}{2}\beta\epsilon\right), r^2 = \lambda^{-1}(1 - x\epsilon), M = \frac{1}{2}\lambda^{-1}\mu\epsilon^2 \qquad \alpha, \beta, \mu \in \mathbb{R} \tag{9}$$

Under a combination of the coordinate transformations (8) and (9), where the limit $\epsilon \downarrow 0$ is taken, our new metric $\tilde{g}$ is defined as:

$$\lambda\tilde{g}(\partial_\tau, \partial_x, \partial_\theta, \partial_\phi, \partial_\psi) = (d\tau + \sigma)^2 + h(\partial_x, \partial_\theta, \partial_\phi, \partial_\psi) \tag{10}$$

where h is defined as:

$$h(\partial_x, \partial_\theta, \partial_\phi, \partial_\psi) = \frac{\rho^2 dx^2}{4\Delta_x} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \frac{\rho^2 d\theta^2}{\Delta_\theta}\left(\frac{\sin^2\theta}{\alpha}d\phi + \frac{\cos^2\theta}{\beta}d\psi\right)^2 + \frac{\Delta_\theta\sin^2\theta\cos^2\theta}{\rho^2}\left(\frac{\alpha - x}{\alpha}d\phi - \frac{\beta - x}{\beta}d\psi\right)^2$$

where we have:

$$\begin{aligned}
\sigma &= \frac{(\alpha - x)\sin^2\theta}{\alpha}d\phi + \frac{(\beta - x)\cos^2\theta}{\beta}d\psi \\
\Delta_x &= x(\alpha - x)(\beta - x) - \mu \\
\rho^2 &= \Delta_\theta - x \\
\Delta_\theta &= \alpha\cos^2\theta + \beta\sin^2\theta
\end{aligned} \tag{11}$$

[11] The introduction of spinors and particular spin bundles will be explained as spinors are encountered in this derivations. Keep reading!

One quick note about why such a transformation requires using an analytic continuation. In physics, one tends to ignore when one is dealing with a spin bundle and when one is dealing with the tangent or cotangent bundles in order to simplify computation. However, in General Relativity, one tends to deal with the standard, real tangent and cotangent bundles of a spacetime manifold M to solve the classical equations of motion. However, when one deals with relativistic quantum mechanics, a spin bundle S is introduced (either implicitly or explicitly) so that solutions to the Dirac equation, which involve spinors $s \in \Gamma(S)$, can be developed. As such, this ad-hoc complexification illustrated above serves as a way to explicitly construct a local trivialization of $S \twoheadrightarrow M$ over some trivializable chart domain $U_\alpha \subset M$.

Using Mathematica, one can quickly establish that the Ricci Tensor for the above metric is related to the $\tilde{g}$ by $\mathrm{Ric} = 4\lambda\tilde{g}$ so this metric is Einstein. A further computation shows that $R = 0$, so that this metric is Ricci-flat. Moreover, one can use an $\mathbb{R}^n$ diffeomorphism to set the free parameter $\mu = 1$ so that the only free parameters are α, β. Now there are a few angle forms in the above metric (the exact ranges aren't established in [10]) and in particular the ψ, ϕ, θ coordinates are periodic (i.e. their chart domains are $[0, k\pi]$, where the choice of k isn't explicit). This generates a $U(1) \times U(1) \times U(1)$ isometry that will help us elucidate the Killing Vectors associated to $\tilde{g}$. Now note that if we regard $\tilde{g}$ as representing a local fibration $U(1) \hookrightarrow U_\alpha \subset M \twoheadrightarrow U_\alpha/U(1)$, where the last quotient is over any of the angle forms ψ, ϕ, θ, we can show that the induced metric on the quotient space represents a Kähler-Einstein manifold with Kähler 2-form $\omega_{\tilde{g}}|_{U_\alpha} = \frac{1}{2}d\sigma$. Let us sketch out the argument in [13]. Let the metric on $N := M/U(1)_\theta$ be denoted $\hat{g}$. Firstly, one considers the quotient over the $U(1)$ fiber corresponding to θ and then uses a computation of the first Chern number to show that there exists a coordinate transformation which reduces to the round metric on $S^2 \times S^2$ with trivial clutching function.[12] Moreover, using the other circle coordinates, we can construct a complete open cover of $(M, \tilde{g})$ (i.e. so that the North Pole, South Pole in the θ coordinate have a non-singular coordinate representation). One can then construct a basis for $H_2(N; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ by looking at the intersection number of chains based in the three different $M/U(1)$ quotients. Finally, we can dualize these chains and get an explicit basis for $H^2(S^2 \times S^2; \mathbb{Z})$ in the coordinates we are looking at. This basis is [13, 23]:

$$
\begin{aligned}
\omega_1 &= \frac{1}{4\pi}\cos\zeta d\zeta \wedge (d\psi - \cos\theta d\phi) + \frac{1}{4\pi}\sin\theta d\theta \wedge d\phi \\
\omega_2 &= \frac{1}{4\pi}\sin\theta d\theta \wedge d\phi
\end{aligned} \tag{12}
$$

Finally, by taking a linear combination of the above basis vectors for $H^2(N; \mathbb{Z})$ and enforcing hermiticity, one arrives at the Kähler form on the quotient: $\omega_{\hat{g}} = \frac{1}{2}d\sigma$.

We have sketched an argument that shows that $(M, \tilde{g})$ is the total space of three $U(1)$ fibrations over a Kähler-Einstein space that is homeomorphic to $S^2 \times S^2$. As it turns out, (10) is in a "standard form" for the local expression of a Sasaki-Einstein metrics so that it is almost automatic that $\tilde{g}$ is Sasaki-Einstein [13]. In the future, this section will include the full derivation of the Killing Vectors associated to $\tilde{g}$ and how the relationship between the Killing Vectors and the constraints on $p, q, r \in \mathbb{Z}$ is established. However, for now, the Killing Vectors and the relationship between the moduli of the Killing Vectors and p, q, r will simply be stated. Firstly note that we have four Killing Vectors for this space:

- Killing Vectors that are compatible with the unsuitability of these coordinates at $\theta = 0, \frac{\pi}{2}$:

$$\partial_\phi, \partial_\psi \tag{13}$$

  This intuitively makes sense, since these vectors vanish when $\theta = 0, \frac{\pi}{2}$ as all the $d\phi, d\psi$ terms in h have $\sin\theta$ in front of them.

- Killing Vectors that are compatible with the unsuitability of these coordinates at the roots of the cubic Δ_x, denoted x_1, x_2, x_3:

$$\ell_i = c_i\partial_\tau + a_i\partial_\phi + b_i\partial_\psi \tag{14}$$

  where $i \in \{1, 2\}$ and,

$$
\begin{aligned}
a_i &= \frac{\alpha c_i}{x_i - \alpha} \\
b_i &= \frac{\beta c_i}{x_i - \beta}
\end{aligned} \tag{15}
$$

$$c_i = \frac{(\alpha - x_i)(\beta - x_i)}{2(\alpha + \beta)x_i - \alpha\beta - 3x_i^2} \tag{16}$$

[12] Recall that a clutching function for S^n is a map $S^{n-1} \to S^n$ that serves at the attaching map for the two n-cells D_1, D_2 that one glues along $\partial D_i \cong S^{n-1}$ to construct S^n. Given a clutching function, one can represent S^n as a CW complex

suppose that $u, v \in \mathbb{Q}$ are such that $0 < v < 1, -v < u < v$. Then we define $p, q, r, s \in \mathbb{Z}$ by [19]:

$$\frac{q-p}{p+q} := \frac{2(v-u)(1+uv)}{4-(1+u^2)(1+v^2)}, \frac{r-s}{p+q} := \frac{2(v+u)(1-uv)}{4-(1+u^2)(1+v^2)}, p+q-r=s \tag{17}$$

These conditions ensure that $\alpha, \beta, \mu, x_1, x_2$ and x_3 are all rational. This is an important consequence of the Chern number restrictions on M. As it turns out, the authors of [13] show that $\frac{l^{-1}}{2\pi}\sigma$ (as per the definition in (10)) is a connection on M if we treat $M \hookrightarrow M/U(1)$ as a $U(1)$ bundle. Since this is independent of the choice of $U(1)$ quotient (the quotient is completely described by the metric h of (10)) and since line bundles over $S^2 \times S^2$ are classified by Chern Classes in $H^2(S^2 \times S^2; \mathbb{Z})$ ([17], §3.1), we have implicit restrictions on the two parameters α, β. Moreover, if we assume that we don't know the diffeomorphism $\mathbb{R}^5 \to \mathbb{R}^5$ that sends μ to unity.[13] Given this disclaimer, here is the relationship between u, v and $\alpha, \beta, \mu, x_1, x_2, x_3, x_3$:

$$\begin{aligned} \alpha &= 1 - \frac{1}{4}(1+u)(1+v), \beta = 1 - \frac{1}{4}(1-u)(1-v), \mu = \frac{1}{16}(1-u^2)(1-v^2) \\ x_1 &= \frac{1}{4}(1+u)(1-v), x_2 = \frac{1}{4}(1-u)(1+v), x_3 = 1 \end{aligned} \tag{18}$$

Finally, we can relate the moduli a_i, b_i, c_i of the Killing Vectors to u, v [19]:

$$\begin{aligned} a_1 &= \frac{(1+v)(3-u-v-uv)}{(v-u)[4-(1+u)(1-v)]} & a_2 &= -\frac{(1+u)(3-u-v-uv)}{(v-u)[4-(1-u)(1+v)]} \\ b_1 &= \frac{(1-u)(3+u+v-uv)}{(v-u)[4-(1+u)(1-v)]} & b_2 &= -\frac{(1-v)(3+u+v-uv)}{(v-u)[4-(1-u)(1+v)]} \\ c_1 &= -\frac{2(1-u)(1+v)}{(v-u)[4-(1+u)(1-v)]} & c_2 &= \frac{2(1+u)(1-v)}{(v-u)[4-(1-u)(1+v)]} \end{aligned} \tag{19}$$

Recall that the goal of this thesis is to study the one-form spectrum of a slightly simpler object, the Sasaki-Einstein metrics $Y^{p,q}$. Given these quite complicated expressions, it will be wise to restrict our initial scope to a specific choice of p, q. The first choice of p, q, $p = q = 1$ is simply the homogeneous metric on $S^2 \times S^3$; this space is known as $T^{1,1}$ and it's spectrum is well-established [15, 27]. The first non-trivial manifold is $Y^{2,1}$ which is an irregular Sasaki-Einstein manifold — that is, the orbits of one of the allowable $U(1)$ actions is dense in $Y^{2,1}$. In general, note that the metric in the form (10) is invariant has $U(1) \times U(1) \times U(1)$ isometries. However, if placed in the original form from [13], one can show that there is actually an *graded* isometry group isomorphic to $\mathsf{SU}(2) \times_{\mathbb{Z}_2} \times U(1) \times U(1)$. For completeness, we present this metric:

$$\begin{aligned} g_{Sparks}(\partial_\theta, \partial_\phi, \partial_y, \partial_\psi, \partial_\alpha) = \frac{(1-cy)}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1}{w(y)q(y)}dy^2 + \frac{q(y)}{9}[d\psi - \cos\theta d\theta]^2 \\ + w(y)\left[d\alpha + \frac{ac - 2y + y^2 c}{6(a-y^2)}[d\psi - \cos\theta d\phi]^2\right] \end{aligned} \tag{20}$$

where we have,

$$\begin{aligned} w(y) &= \frac{2(a-y^2)}{1-cy} \\ q(y) &= \frac{a - 3y^2 + 2cy^3}{a - y^2} \end{aligned}$$
$$0 \le \theta \le \pi, \quad 0 \le \phi, \psi \le 2\pi, \quad y_1 \le y \le y_2, \quad y < 1, \quad 0 \le \alpha \le 2\pi\ell$$

where y_1, y_2 are roots of the cubic $q(y)$ and due to symmetry considerations $\ell = \frac{q}{3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}}$. For reasons explained on page 3 of [13], we are forced to restrict $a \in (0, 1)$. From this definition, it is apparent that ψ, α have $U(1)$ isometries. However, note that one can rewrite the expressions for θ, ϕ in terms of the round metric on S^3, so that θ has an $\mathsf{SU}(2)$ isometry. When one rewrites (20) in terms of the Maurer-Cartan forms of $\mathsf{SU}(2)$, it is apparent that there is a $U(1)$ right action (from the Killing Vector ∂_ϕ) and an $\mathsf{SU}(2)$ left action (derived from the Killing Vector ∂_θ).

[13] This is a good assumption, at least based on the scant efforts at numerical computation related to these manifolds. For instance in the Master's Thesis [19], the author notes that he could not find a closed form diffeomorphism (or even an approximation) using Mathematica.

4.3 The Scalar Laplacian

The Scalar Laplacian (i.e. the Laplacian-Beltrami Operator on Functions $C^\infty(M)$) of the $Y^{p,q}$ manifolds has been studied in [19, 20, 25]. Unfortunately, we have two metrics to deal with, (10), (20). The solutions of (20) are a bit harder to elucidate, but they will be discussed in this section. From §4.3.1 onwards, we will solely use (10) simply because it is easier to work with. Using separation of variables, the authors were able to deduce that harmonic functions associated to this operator (up to a scalar multiple) for (20) are of the form [20],

$$\Psi_{Sparks}(y,\theta,\phi,\psi,\alpha) = \exp\left(i\left[N_\phi\phi + N_\psi\psi + \frac{N_\alpha}{\ell}\alpha\right]\right)R(y)\Theta(\theta) \tag{21}$$

The expressions for $R(y), \Theta(\theta)$ are a bit complicated and require a more delicate analysis than the angular solutions. The authors make many analogies to the solutions of the non-relativistic, time-independent, Schrödinger Equation and consider $R(y)$ to be the "radial" function for $Y^{p,q}$ and $\Theta(\theta)$ to be the "angular" function for $Y^{p,q}$. These analogies are quite valid since we are dealing with a manifold that is diffeomorphic to $S^2 \times S^3$ and we can rewrite our metric in terms of the Maurer-Cartan forms on S^3. In fact, the Θ equation, while not explicitly known, is an eigenfunction of the Casimir operator $\hat{K}$ of $\mathsf{SU}(2)$, which has the coordinate expression:

$$\hat{K} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\left(\frac{\partial}{\partial\phi} + \cos\theta\frac{\partial}{\partial\psi}\right)^2 + \left(\frac{\partial}{\partial\psi}\right)^2 \tag{22}$$

and as per the physics convention we have $\hat{K}\Theta(\theta) = -L(L+1)\Theta(\theta), L \in Z$. It turns out that the ordinary differential equation for $R(y)$ obtained via separation of variables is in fact *Heun's equation*, an equation of Fuchsian-type with four regular singularities at $y = y_1, y_2, y_3, \infty$. The explicit solution will be discussed in the next section after we write the Ordinary Differential Equations for each variable.

4.3.1 The Scalar Laplacian in Coordinates

Without further ado, the Scalar Laplacian $\Delta_{(5)}$ in coordinates (with regard to the metric (10)) is [19]:

$$\begin{aligned}\Delta_{(5)} \to\ & \frac{4}{\rho^2}\frac{\partial}{\partial x}\left(\Delta_x\frac{\partial}{\partial x}\right) + \frac{4}{\rho^2}\frac{\partial}{\partial y}\left(\Delta_y\frac{\partial}{\partial y}\right) + \frac{\partial^2}{\partial\tau^2} \\ &+ \frac{\alpha^2\beta^2}{\rho^2\Delta_x}\left(\frac{(\beta-x)}{\beta}\frac{\partial}{\partial\phi} + \frac{(\alpha-x)}{\alpha}\frac{\partial}{\partial\psi} - \frac{(\alpha-x)(\beta-x)}{\alpha\beta}\frac{\partial}{\partial\tau}\right)^2 \\ &+ \frac{\alpha^2\beta^2}{\rho^2\Delta_y}\left(\frac{(1+y)}{\beta}\frac{\partial}{\partial\phi} - \frac{(1-y)}{\alpha}\frac{\partial}{\partial\psi} - \frac{(\alpha-\beta)(1-y^2)}{2\alpha\beta}\frac{\partial}{\partial\tau}\right)^2\end{aligned} \tag{23}$$

where $\Delta_y := (1-y^2)\Delta_\theta$.

While these coordinates are slightly different that those in , the only change in the resulting eigenfunction will arise in different normalization constants (i.e. the process of defining Ψ so that $\|\Psi\|_{L^2(M)} = 1$) with the eigenfunctions of (23) having normalization constants that depend on α, β, ρ as opposed to a and c. Note that we can simplify (23) significantly if we express Δ_y, Δ_x in terms of their roots. That is, if x_1, x_2, x_3 are the roots of Δ_x and y_1, y_2, y_3 are the roots of Δ_y (see (11)), the metric becomes [19]:

$$\begin{aligned}\Delta_{(5)} \to\ & \frac{\partial^2}{\partial\tau^2} + \frac{4}{\rho^2}\frac{\partial}{\partial x}\left(\Delta_x\frac{\partial}{\partial x}\right) + \frac{\Delta_x}{\rho^2}\left(\frac{1}{(x-x_1)}v_1 + \frac{1}{(x-x_2)}v_3 + \frac{1}{(x-x_3)}v_5\right)^2 \\ &+ \frac{4}{\rho^2}\frac{\partial}{\partial y}\left(\Delta_y\frac{\partial}{\partial y}\right) + \frac{\Delta_y}{\rho^2}\left(\frac{1}{(y-y_1)}v_2 + \frac{1}{(y-y_2)}v_4 + \frac{1}{(y-y_3)}v_6\right)^2\end{aligned} \tag{24}$$

The separable solutions in these coordinates are quite similar to those in (4.3.1) except that we now have lost the angular function Θ and we have replaced it with another solution to a Heun's Differential Equation. Explicitly, we have the eigenfunction,

$$\Psi(\tau,\phi,\psi,x,y) = \exp\left(i\left[N_\tau\tau + N_\phi\phi + N_\psi\psi\right]\right)F(x)G(y) \tag{25}$$

where F, G are defined by the *Heun's Differential Equations*,

$$\frac{d^2F}{dx^2} + \left(\frac{1}{(x-x_1)} + \frac{1}{(x-x_2)} + \frac{1}{(x-x_3)}\right)\frac{dF}{dx} + Q_xF = 0 \tag{26}$$

$$\frac{d^2G}{dy^2} + \left(\frac{1}{(y-y_1)} + \frac{1}{(y-y_2)} + \frac{1}{(y-y_3)}\right)\frac{dF}{dy} + Q_yF = 0 \tag{27}$$

where we have,

$$Q_x = \frac{1}{\Delta_x}\left(\mu_x - \frac{1}{4}Ex - \sum_{i=1}^{3}\frac{\alpha_i^2}{x-x_i}\frac{d\Delta_x}{dx}(x_i)\right), \quad Q_y = \frac{1}{H_y}\left(\mu_y - \frac{1}{4}Ey - \sum_{i=1}^{3}\frac{\beta_i^2}{y-y_i}\frac{dG}{dy}(y_i)\right) \tag{28}$$

$$\alpha_i = -\frac{1}{2}(a_i N_\phi + b_i N_\psi + c_i N_\tau), \quad \beta_1 = \frac{1}{2}N_\phi,\ \beta_2 = \frac{1}{2}N_\psi,\ \beta_3 = \frac{1}{2}(N_\tau - N_\phi - N_\psi) \tag{29}$$

$$H_y = (y-y_1)(y-y_2)(y-y_3), \quad \mu_x = \frac{1}{4}C - \frac{1}{2}N_\tau(\alpha N_\phi + \beta N_\psi) + \frac{1}{4}(\alpha+\beta)N_\tau^2 \tag{30}$$

$$\mu_y = \frac{1}{2(\beta-\alpha)}\left(-C + \left(\frac{\alpha+\beta}{2}\right)E + 2(\alpha N_\phi + \beta N_\psi)N_\tau - (\alpha+\beta)N_\tau^2\right) \tag{31}$$

To do:

- Explain Heun's Differential Equation
- Show the construction of the Heun Function

4.3.2 The Spectrum of the Scalar Laplacian

To be completed — Requires the Heun Function

4.4 The Hodge-de Rham Laplacian for One-Forms in coordinates

The final piece of information that we need to state the problem is the Hodge-de Rham Laplacian on n-forms, $\Delta^n_{HdR} = d_{n+1}\delta_{n+1} + \delta_n d_n$, where d_i is the exterior derivative on i-forms and δ_i is the formal adjoint (often times denotes d^*) of d under the inner product induced by the Hodge Star operator. Due to the unwieldiness of the above equations, we will only consider Δ^1_{HdR}.

5 Problem Statement

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